Robust Fitting

Mathematical Models and Methods for Image Processing

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Fitting geometric primitives is ubiquitous in Computer Vision

Example: Line Fitting for Vanishing Point Estimation

x

$ax + by + c = 0$

$\theta = (a, b, c)$

Example: Conic Fitting

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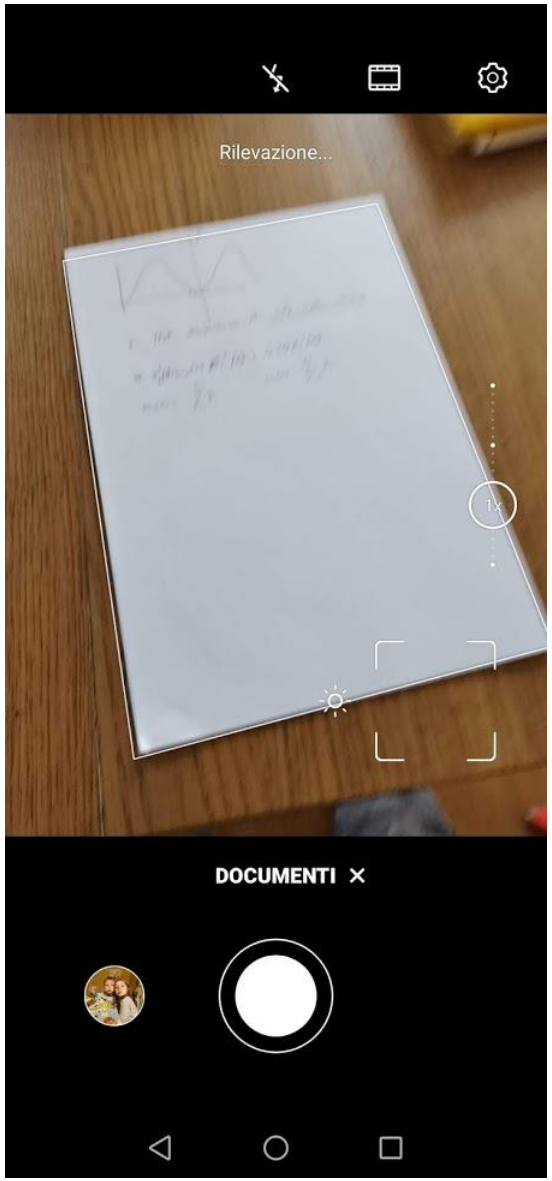
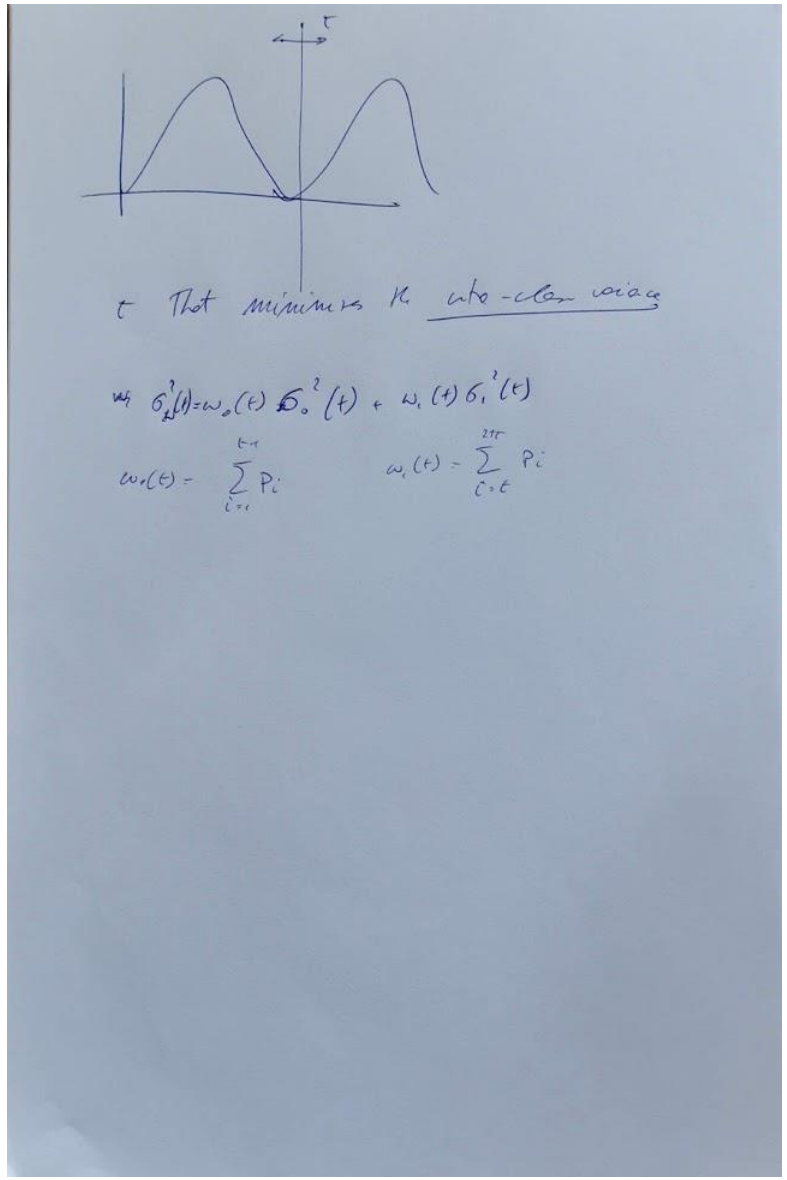
Line Detection is Important





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MATHEMATICS FOR INNOVATION

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Example: Estimating Homographic Transformations

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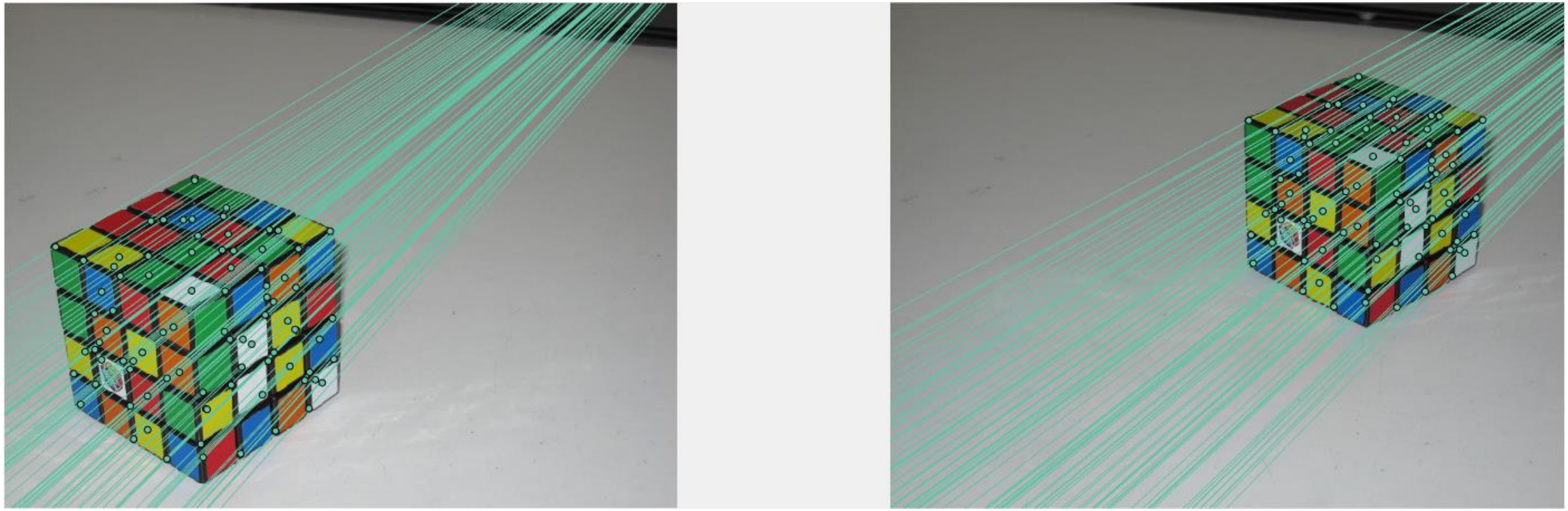
Example: Estimating Fundamental Matrix



In all these cases the problem boils down to fitting a parametric model to (presumably) noisy input data

Ordinary Least Square

All these problems boils down to..

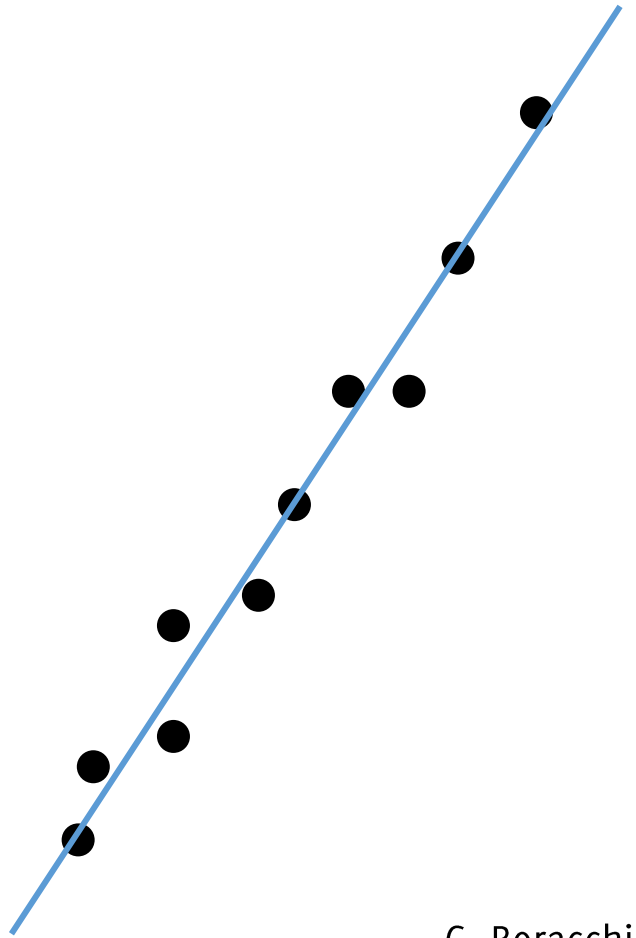
Given a set of N points (or matches..)

$$X = \{(x_1, y_1), \dots, (x_N, y_N)\}$$

Given a parametric model

$$y = mx + q$$

Estimate the parameters m, q yielding the **best fit**



Least Square Regression

Given a set of N points (or matches..)

$$X = \{(x_1, y_1), \dots, (x_N, y_N)\}$$

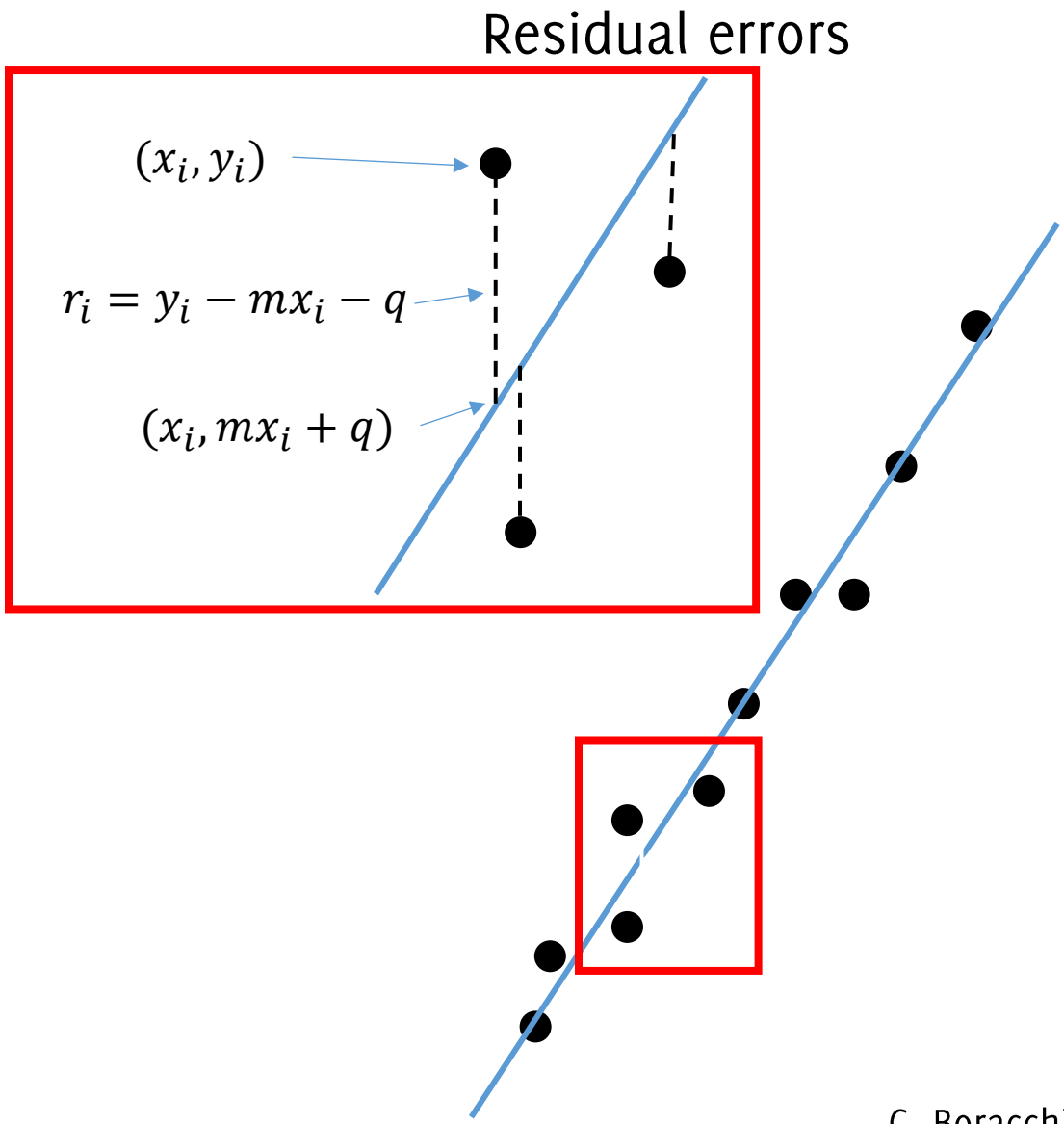
Given a parametric model

$$y = mx + q$$

Estimate the parameters m, q yielding the **best fit**

The best fit is the one **minimizing some residual error over the whole data**

$$r_i = y_i - mx_i - q$$



Ordinary Least Square (OLS) Regression

The loss function is the sum of squared residual errors

$$E = \sum_{i=1}^{N} (r_i)^2 = \sum_{i=1}^{N} (y_i - mx_i - q)^2 =$$

Which in matrix form becomes

$$r_i = y_i - [x_i \;\; 1] \begin{bmatrix} m \\ q \end{bmatrix}, \qquad i = 1, \dots, N$$

$$E = \left\| \begin{bmatrix} y_1 \\ \dots \\ y_N \end{bmatrix} - \begin{bmatrix} x_1 & 1 \\ \dots & \\ x_N & 1 \end{bmatrix} \begin{bmatrix} m \\ q \end{bmatrix} \right\|_2^2$$

Then, the solution can be computed via least square regression

$$[\hat{m}, \hat{q}] = \underset{m,q}{\operatorname{argmin}} \left\| Y - X \begin{bmatrix} m \\ q \end{bmatrix} \right\|_2^2$$

Ordinary Least Square (OLS) Regression

OLS consists in solving the following problem

$$[\hat{m}, \hat{q}] = \underset{m,q}{\operatorname{argmin}} \frac{1}{2} \left\| Y - X \begin{bmatrix} m \\ q \end{bmatrix} \right\|_2^2$$

by zeroing the derivative of the loss function

$$\frac{\partial}{\partial \theta} \frac{1}{2} \|Y - X\theta\|_2^2 = 0, \qquad \theta = \begin{bmatrix} m \\ q \end{bmatrix}$$

This follows from matrix calculus

$$\frac{\partial}{\partial \theta} \frac{1}{2} \|Y - X\theta\|_2^2 = \frac{1}{2} 2X^\intercal (X\theta - Y)$$

Thus the solutoin becomes

$$X^\intercal \left(X\hat{\theta} - Y \right) = 0 \rightarrow \hat{\theta} = (X^\intercal X)^{-1} X^\intercal Y$$

This error does not make sense
when the line is vertical

What about minimizing point-line distance?

Given a set of N points (or matches..)

$$X = \{(x_1, y_1), \dots, (x_N, y_N)\}$$

Given a parametric model

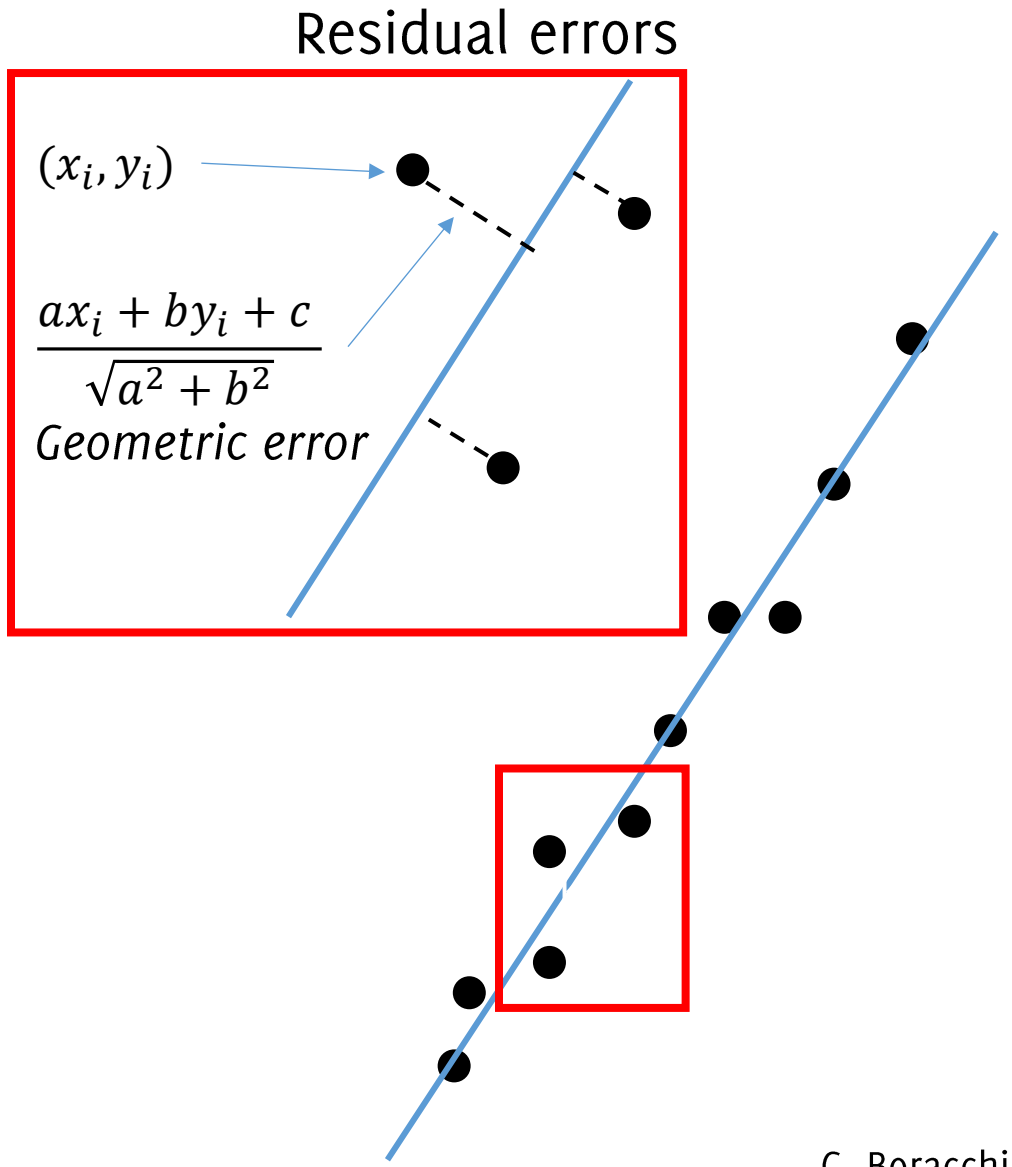
$$ax + by + c = 0$$

Estimate the line parameters a, b, c yielding the **best fit minimizing** the **algebraic error**

$$E = \sum_{i=1}^{N} (ax_i + by_i + c)^2$$

If we take

$$r_i = ax_i + by_i + c$$



What about minimizing the algebraic error?

Given a set of N points (or matches..)

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$$E = \sum_{i=1}^{N} (ax_i + by_i + c)^2$$

This is not the geometric error, as it has no denominator $\sqrt{a^2 + b^2}$.
The geometric error does not admit a linear expression to be optimized, however, this error does not have the «issues with vertical lines» as the previous one

What about minimizing point-line distance?

$$E = \left\| \begin{bmatrix} x_1 & y_1 & 1 \\ & \dots & \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\|_2^2$$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \|A\theta\|_2^2,$$

Being the parameter vector $\theta = [a; b; c]$ and constraining $\|\theta\|_2 = 1$ due to the equivalence relation between θ and lines

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \|A\theta\|_2^2, \text{subject to } \|\theta\|_2 = 1$$

The DLT solves this problem by minimizing the algebraic error!

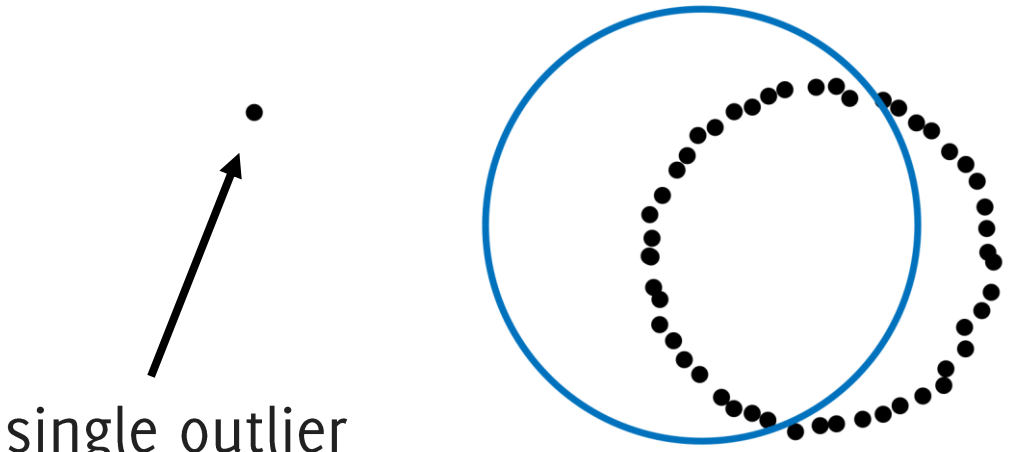
$\theta = V(:, \text{end})$, being $A = UDV^{\mathsf{T}}$

Robustness to Outliers

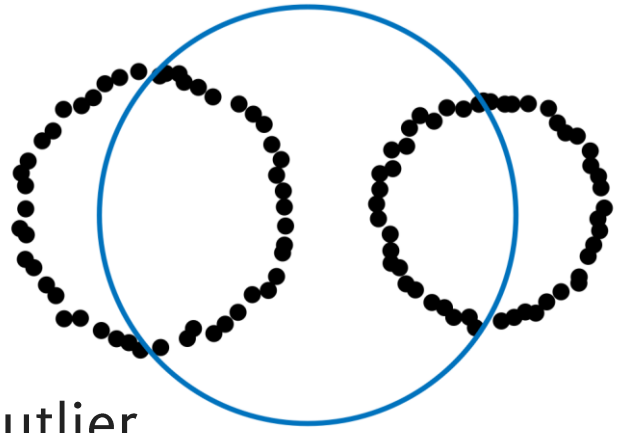
Least squares breaks down

Break down point: the proportion of incorrect observations that can be handled before giving an arbitrarily large uncorrect result.

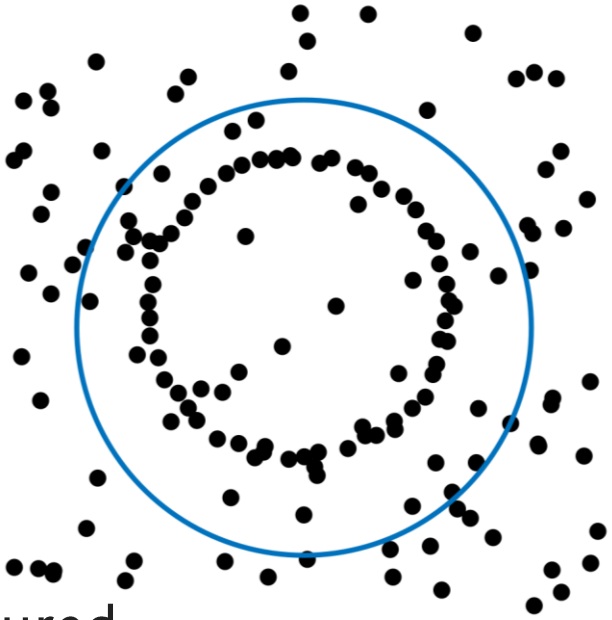
Least squares has 0% breakdown point (the outlier might be arbitrarily large, i.e., $\rightarrow \infty$)



single outlier



pseudo-outlier



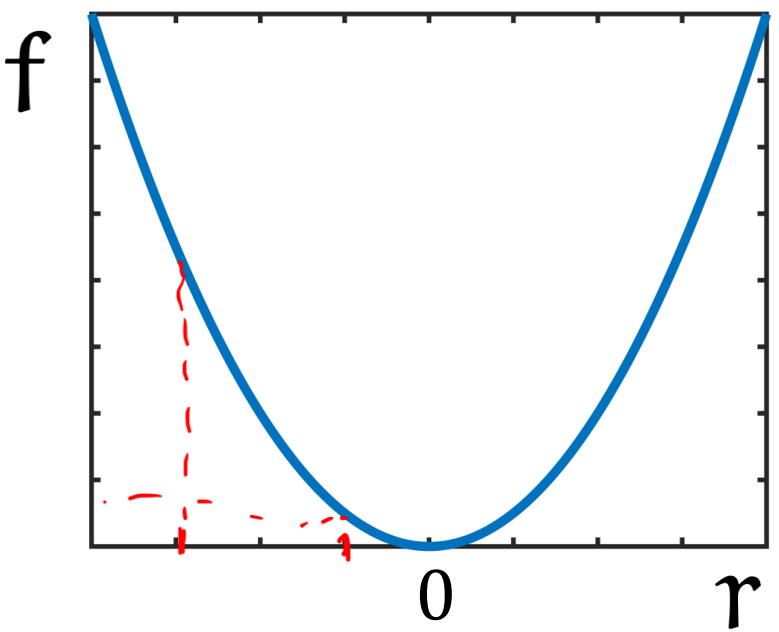
unstructured outliers

The loss function in OLS

The loss so far is the sum of a function of all the residuals

$$E = \sum_{i=1}^{N} f(r_i), \text{where } f(r_i) = r_i^2$$

However, other options for f are viable, giving rise to different loss functions, and different results

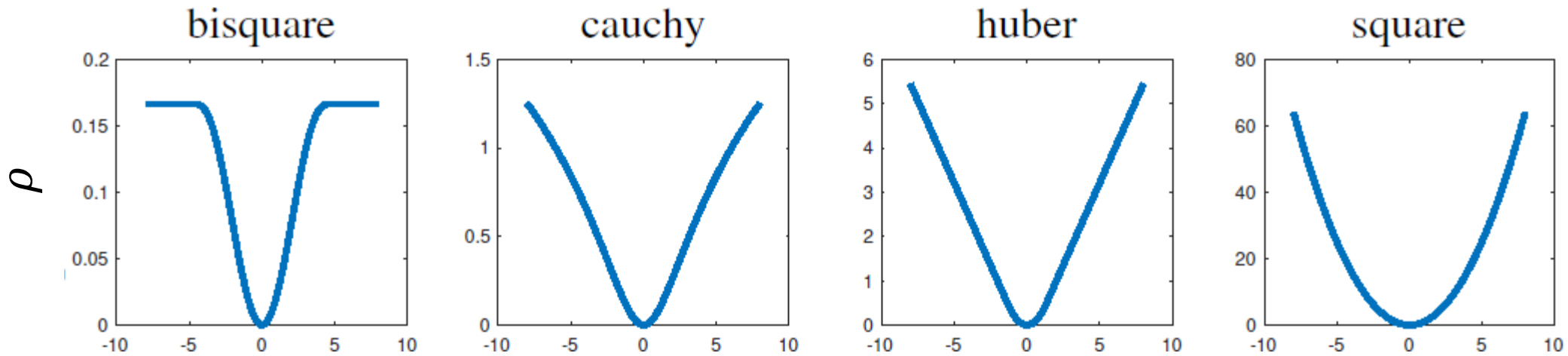


M-Estimator

Replaces the squared loss in the OLS with a different loss function which penalizes less large residual values (deemed to correspond to outliers)

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{N} \rho(r_i(\theta))$$

Where ρ a symmetric, positive-definite function having a unique minimum in zero



Andrea Fusiello, Elaborazione delle Immagini: Visione Computazionale, http://www.diegm.uniud.it/fusiello/index.php/Visione_Computazionale

M-Estimator

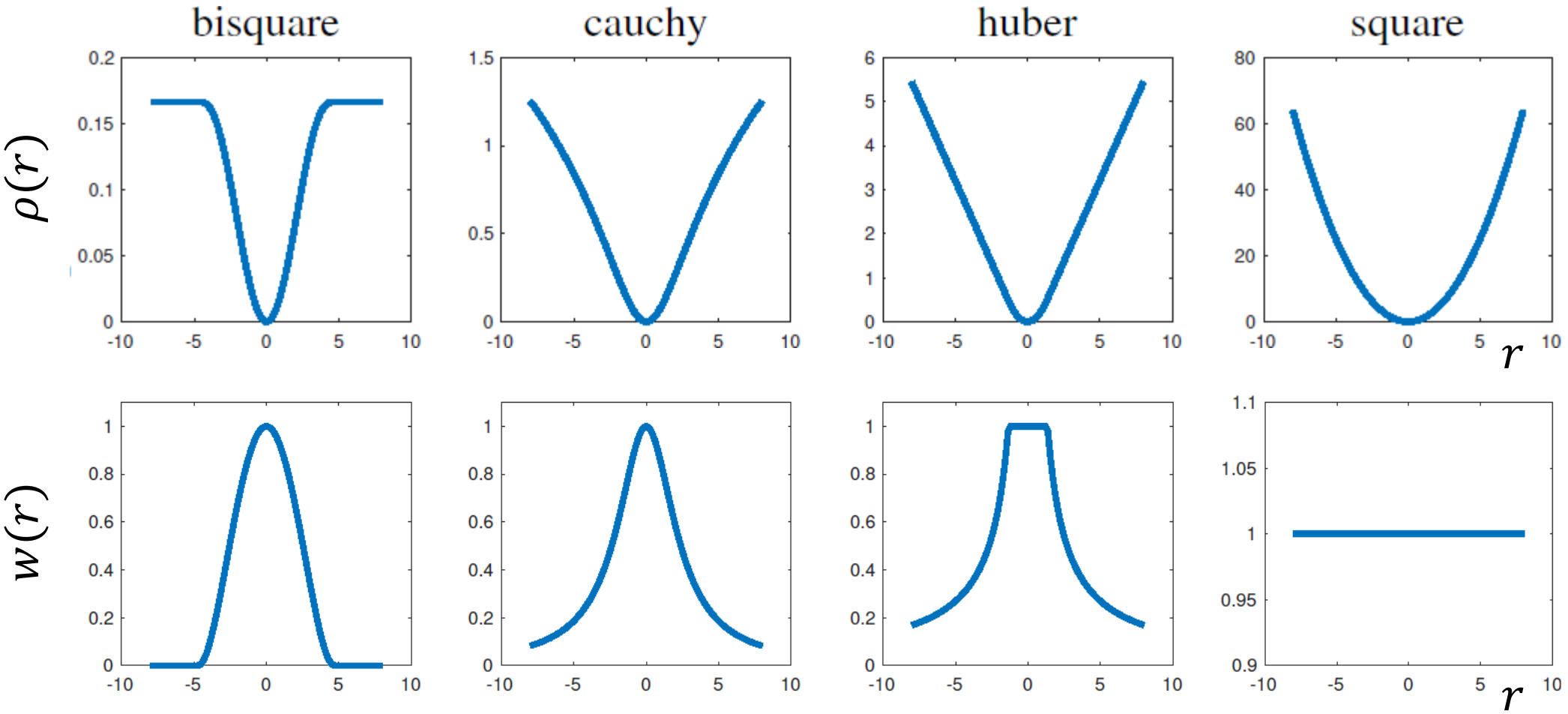
To solve this minimization problem we need to zero the derivative for each dimension of θ

$$\sum_{i=1}^{N} \rho'(r_i) \frac{\partial r_i}{\partial \theta_j} = 0, \qquad j = 1, \dots, M$$

This problem is solved by an iterative reweighted least square

M-Estimator

Weights associated to the previous losses are



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RanSaC

Robust single model fitting: consensus maximisation

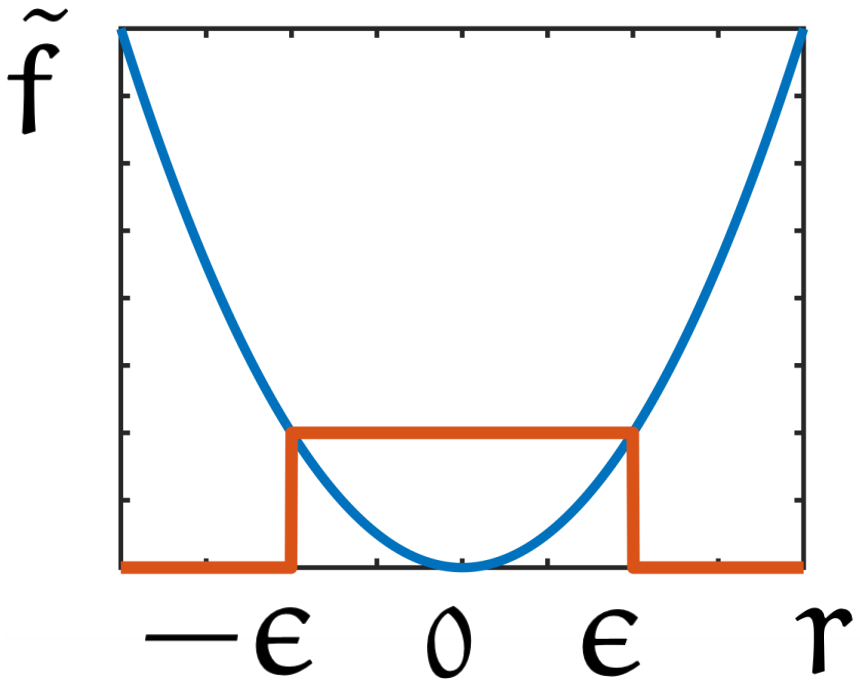
Instead of minimizing the residuals, **maximize the consensus.** Define:

- an inlier threshold $\epsilon > 0$
- a consensus function $\tilde{f}$ which is

$$\tilde{f}(r_i) = \begin{cases} 1, & r_i \leq \epsilon \\ 0, & r_i > \epsilon \end{cases}$$

Identify $\hat{\theta}$ that maximizes tha consensus

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{N} \tilde{f}\left(r_i\left(x_i, \theta\right)\right)$$



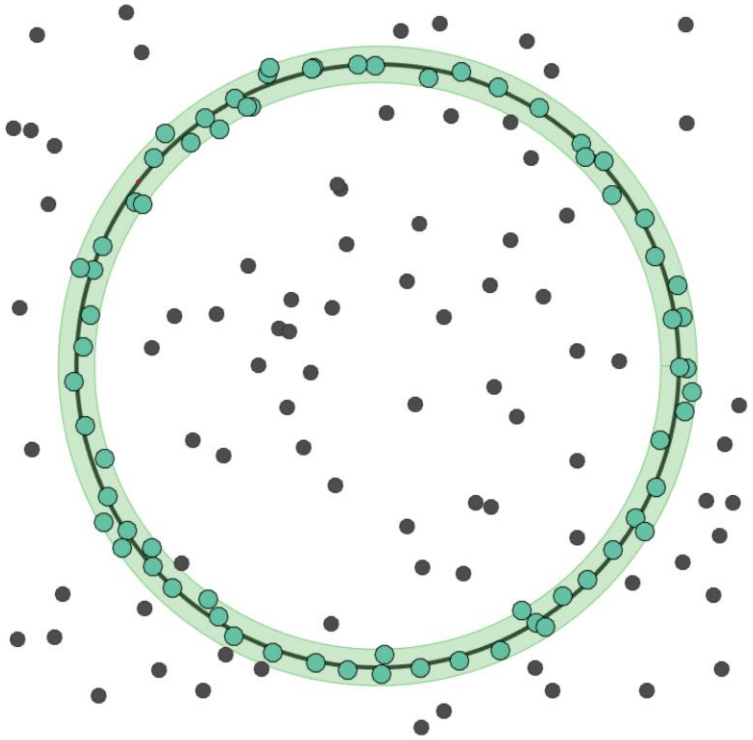
The Conensus Set

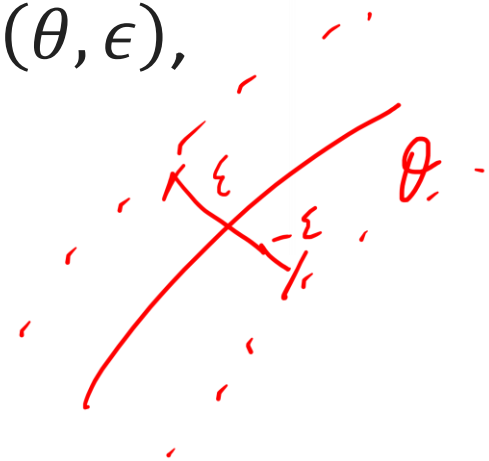
Consensus set

$$\mathrm{CS}(\theta, \epsilon) = \{x_i \mid r_i \leq \epsilon\}$$

Being $r_i = r(x_i, \theta)$, the residual of the model θ at a point x_i

The larger the consensus set $\mathrm{CS}(\theta, \epsilon)$, the better the model θ





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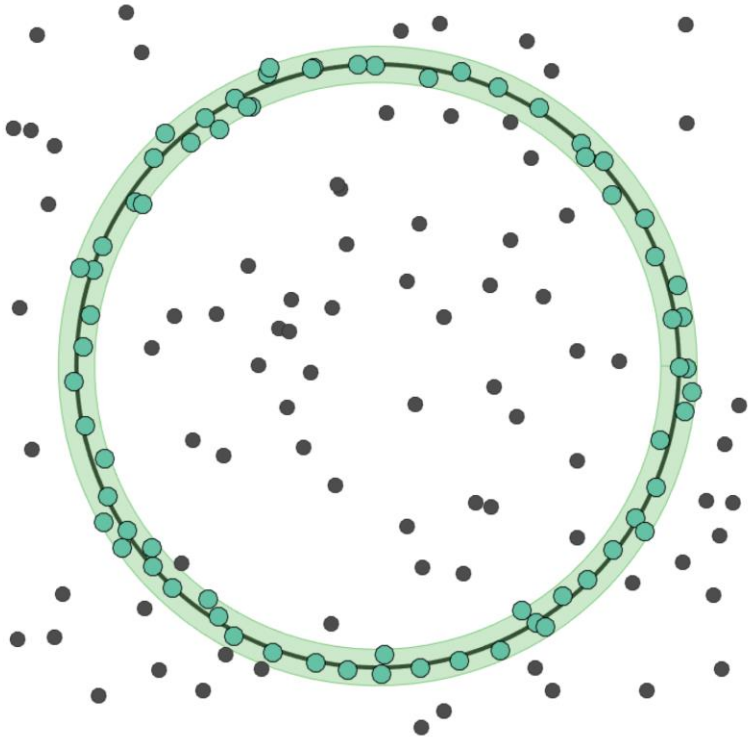
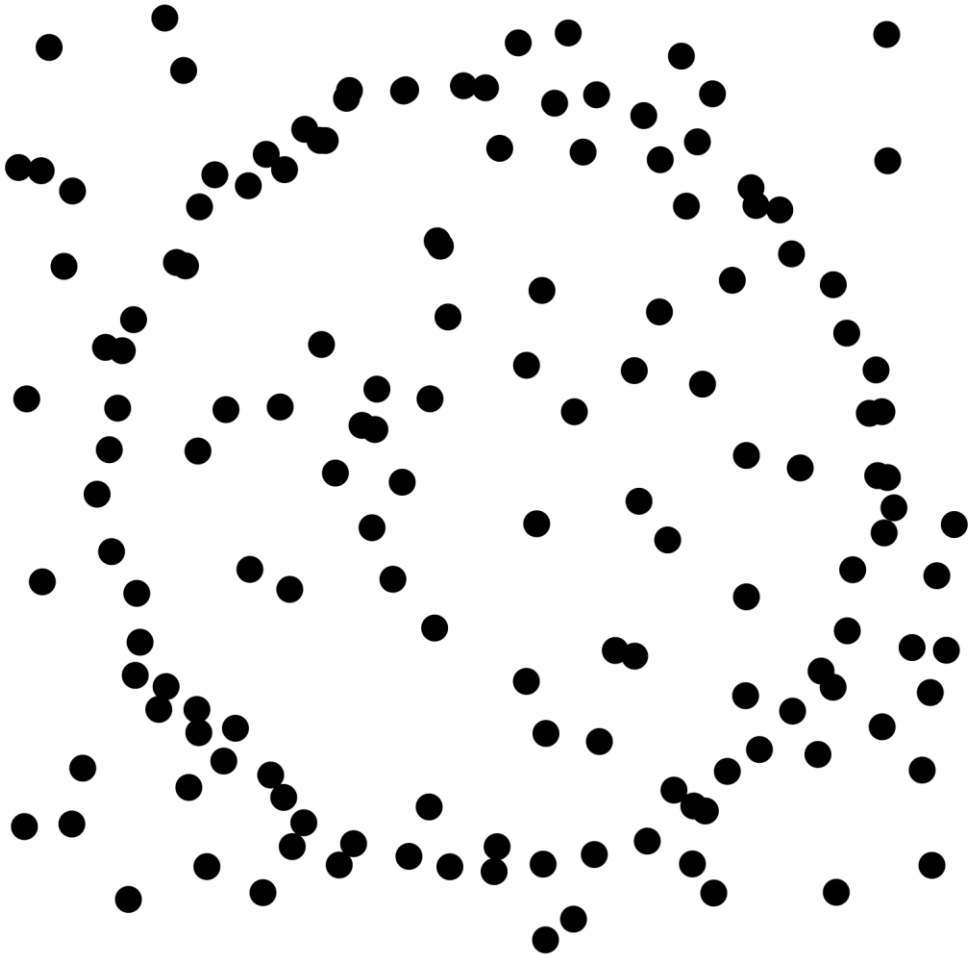


We have been fitting lines so far, but **everything** holds for any parametric model (e.g. circle, conics, homographies, fundamental matrices)

Randomized Sample Consensus [Fischler and Bolles 1981]



Input: X data, ϵ inlier threshold, k_{max} max iteration
Output: θ^* model estimate
$J^* = -\infty, k = 0;$
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \sum_{x \in X} \hat{f}_\epsilon(r(x, \theta))$;
  if $J(\theta) > J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

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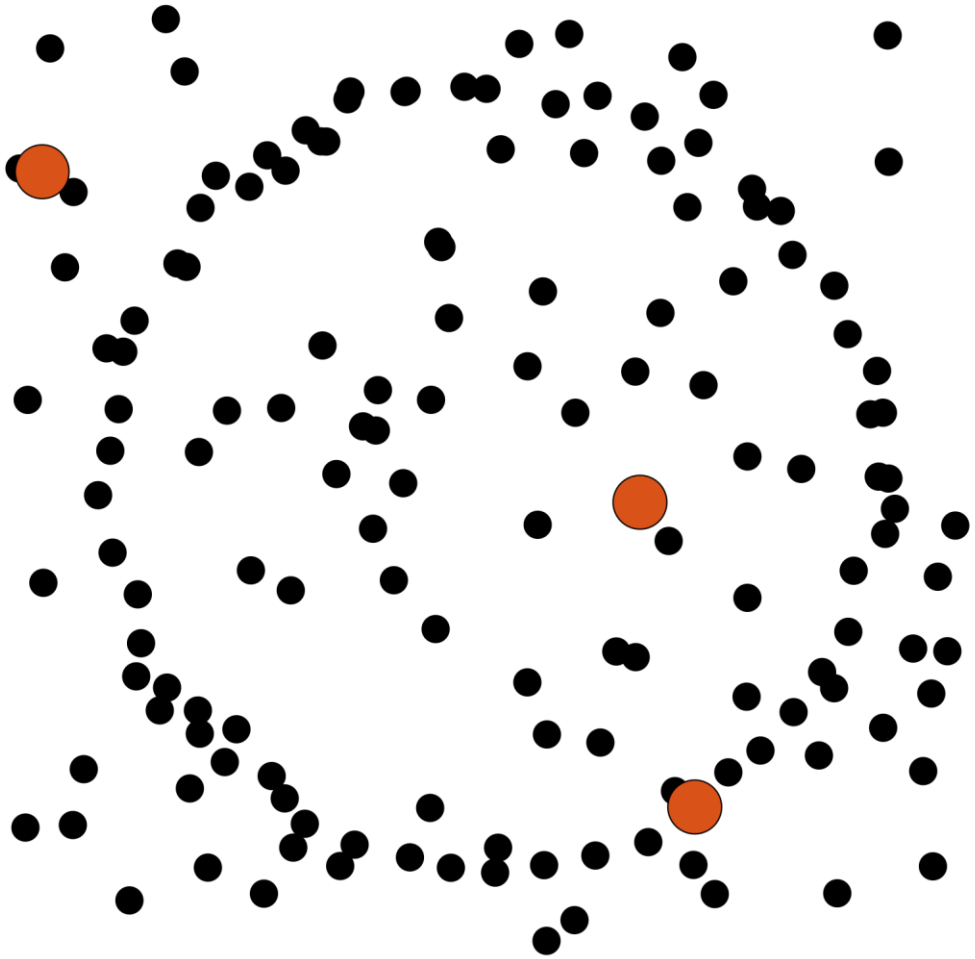

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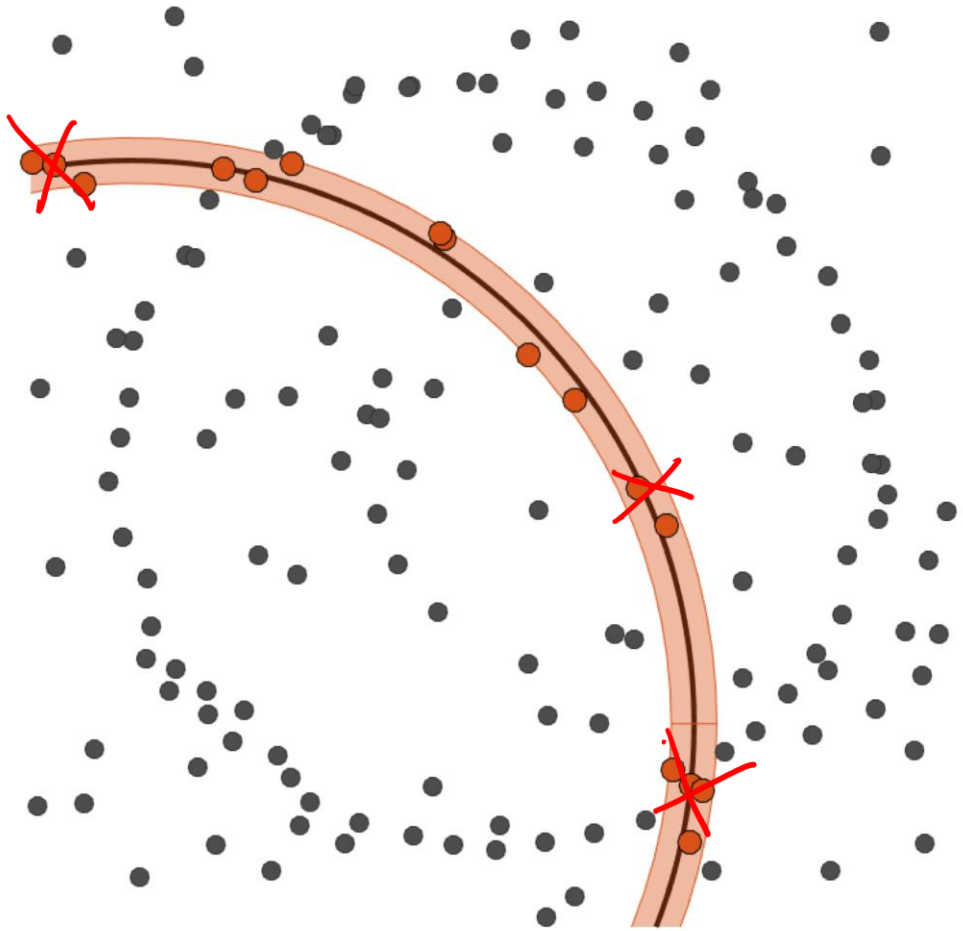

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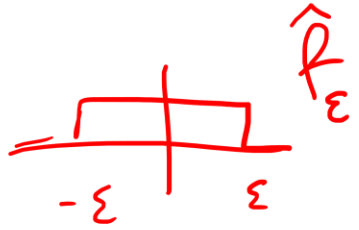

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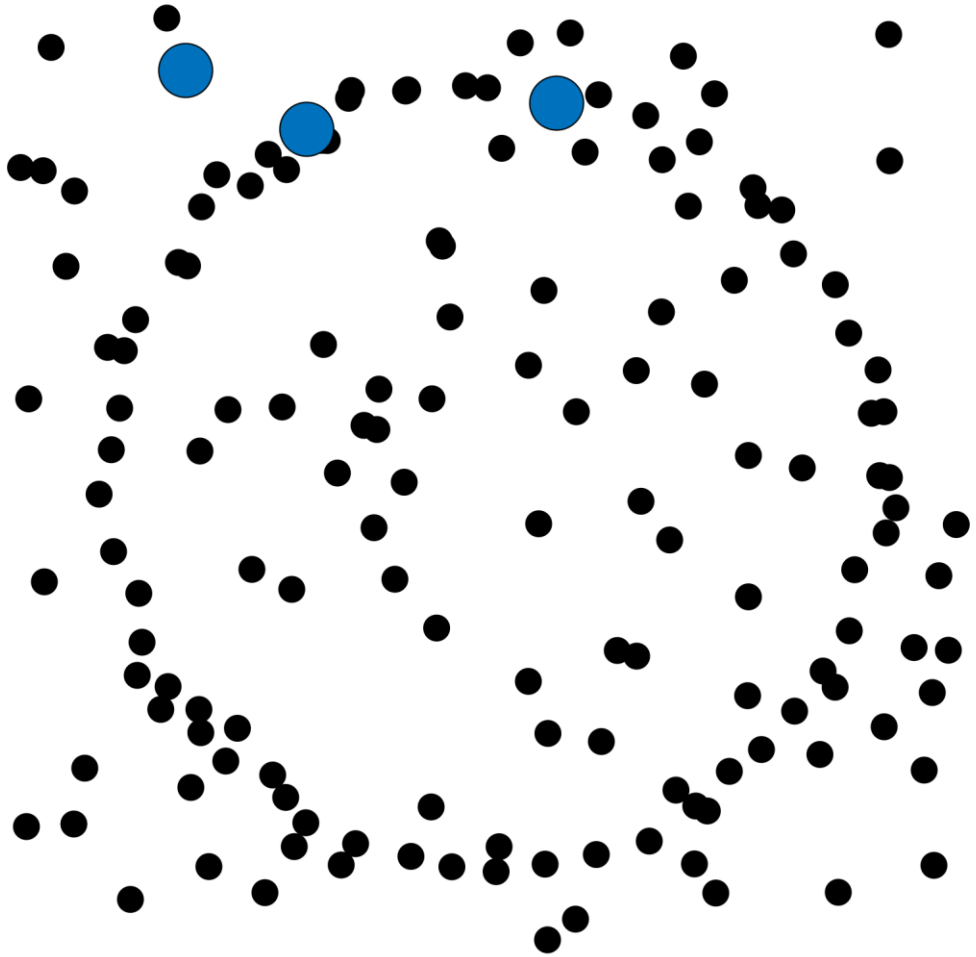

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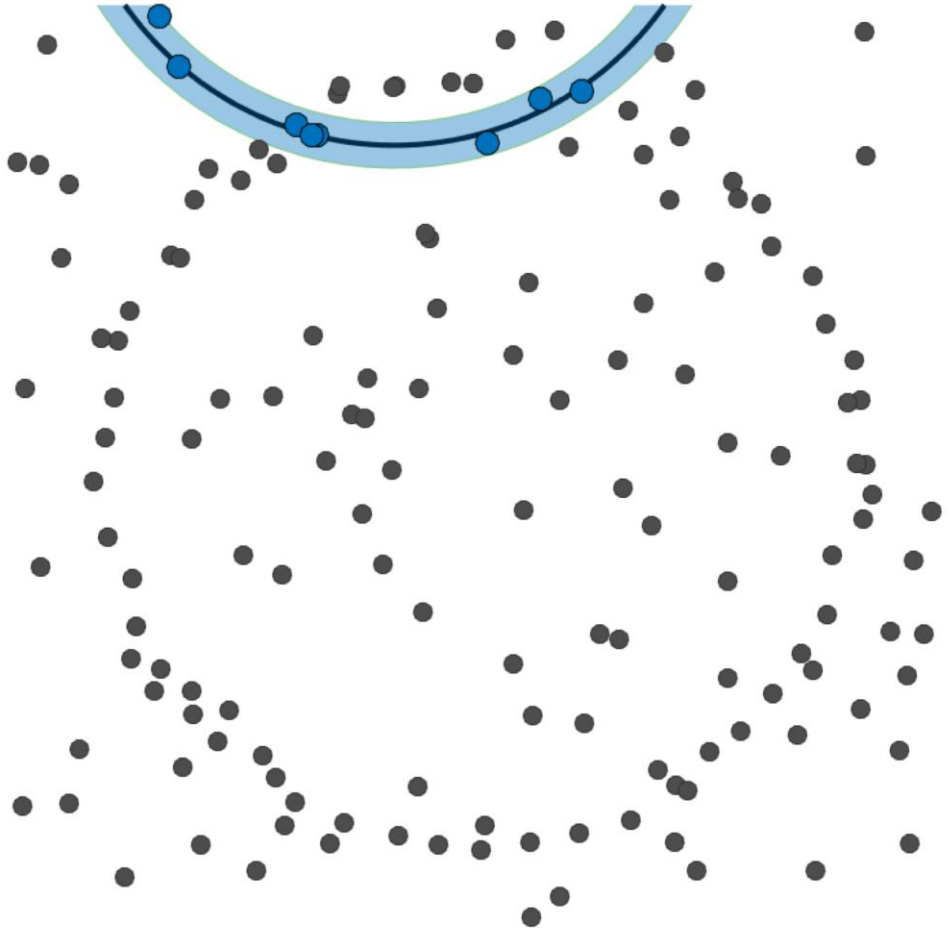

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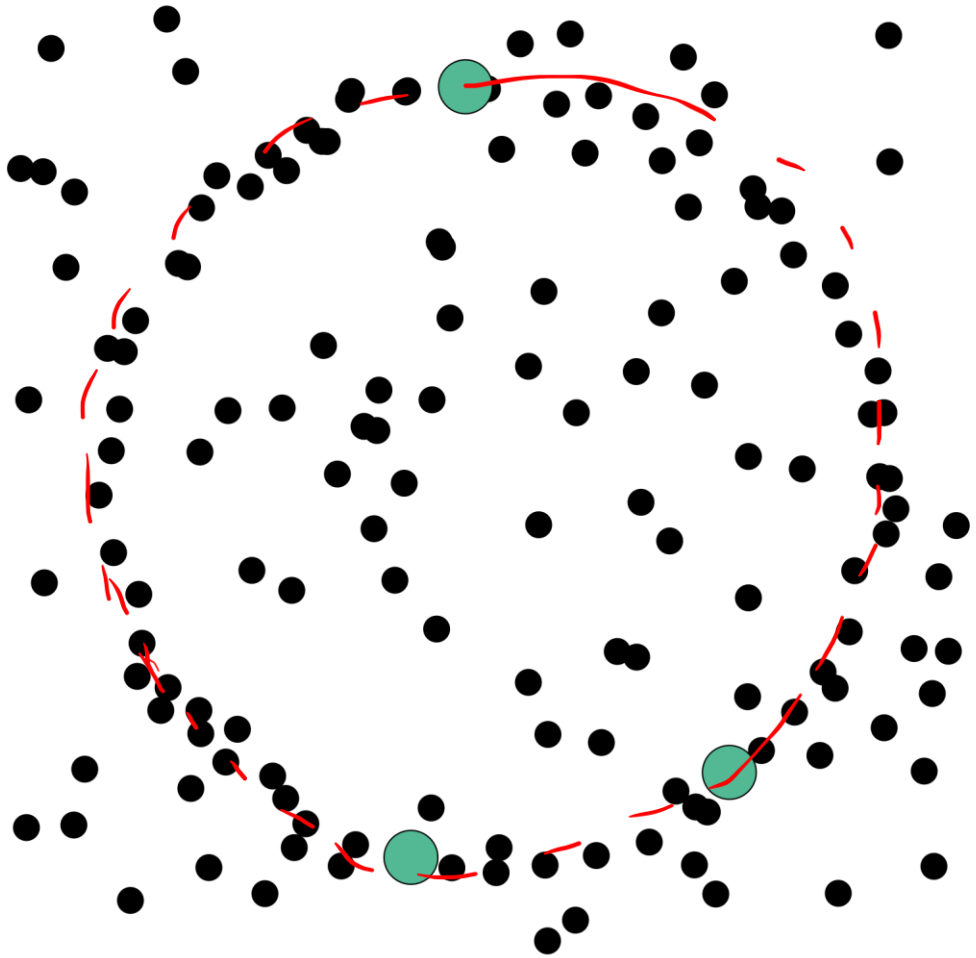

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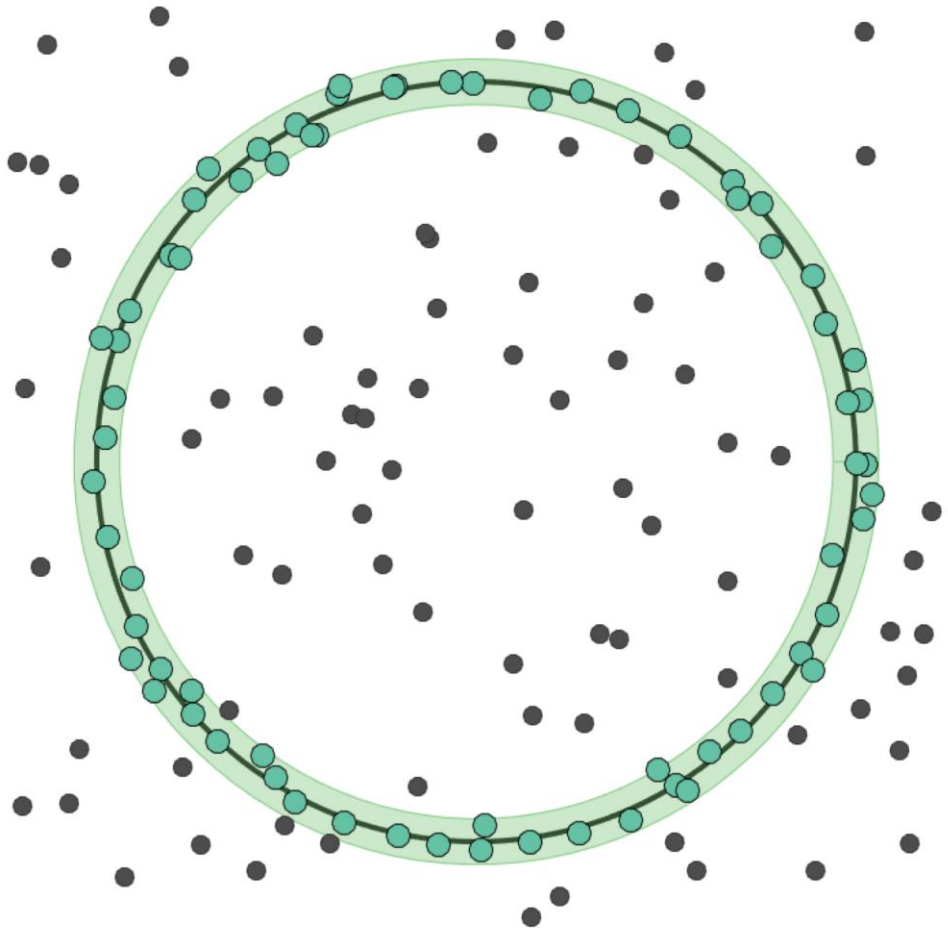

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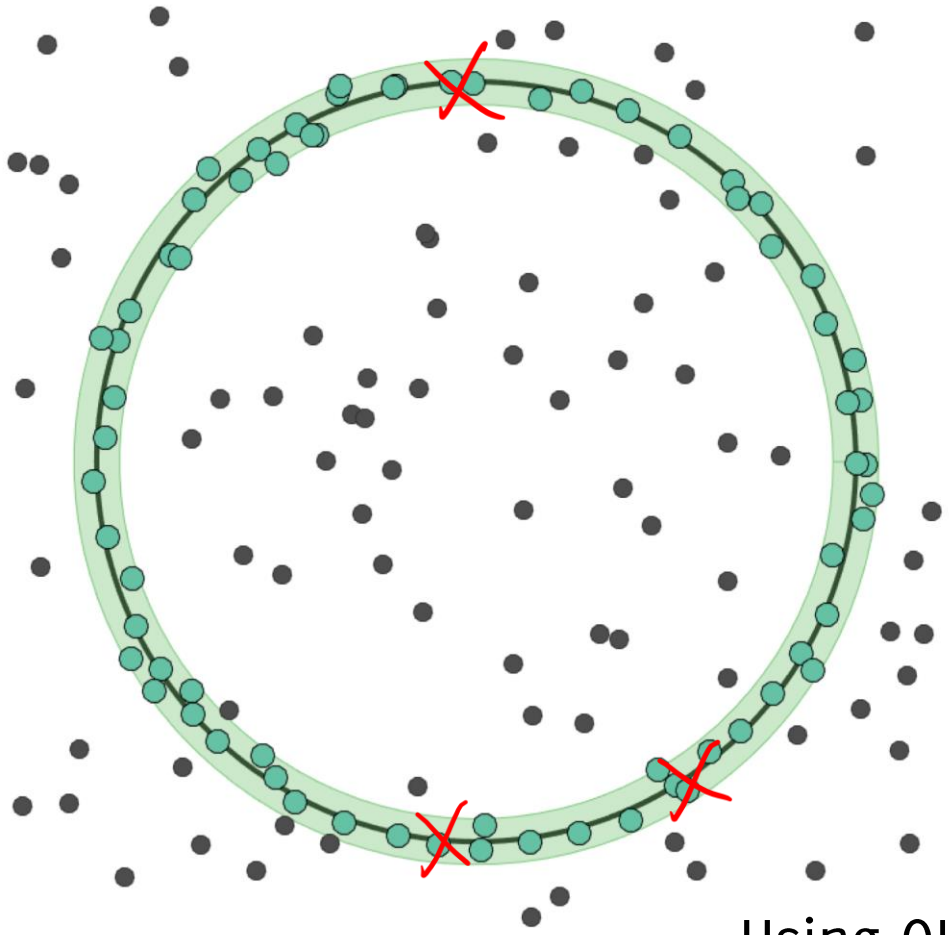

Using OLS,
Increase stability of

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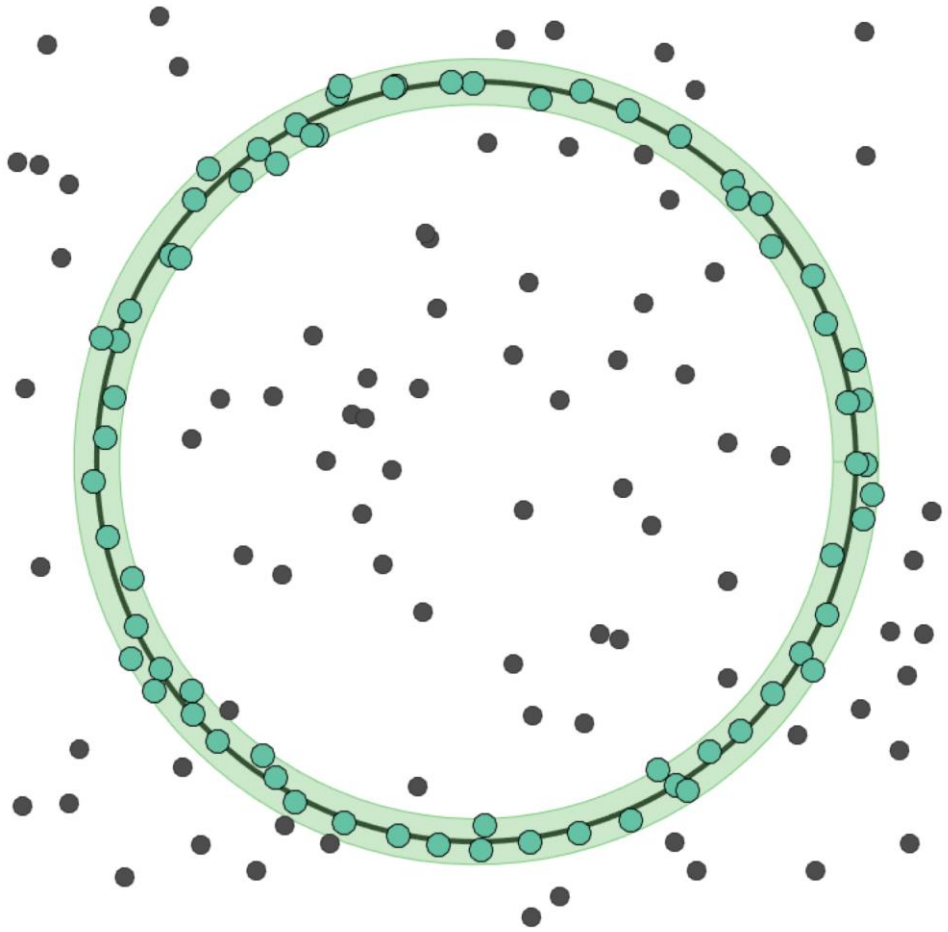
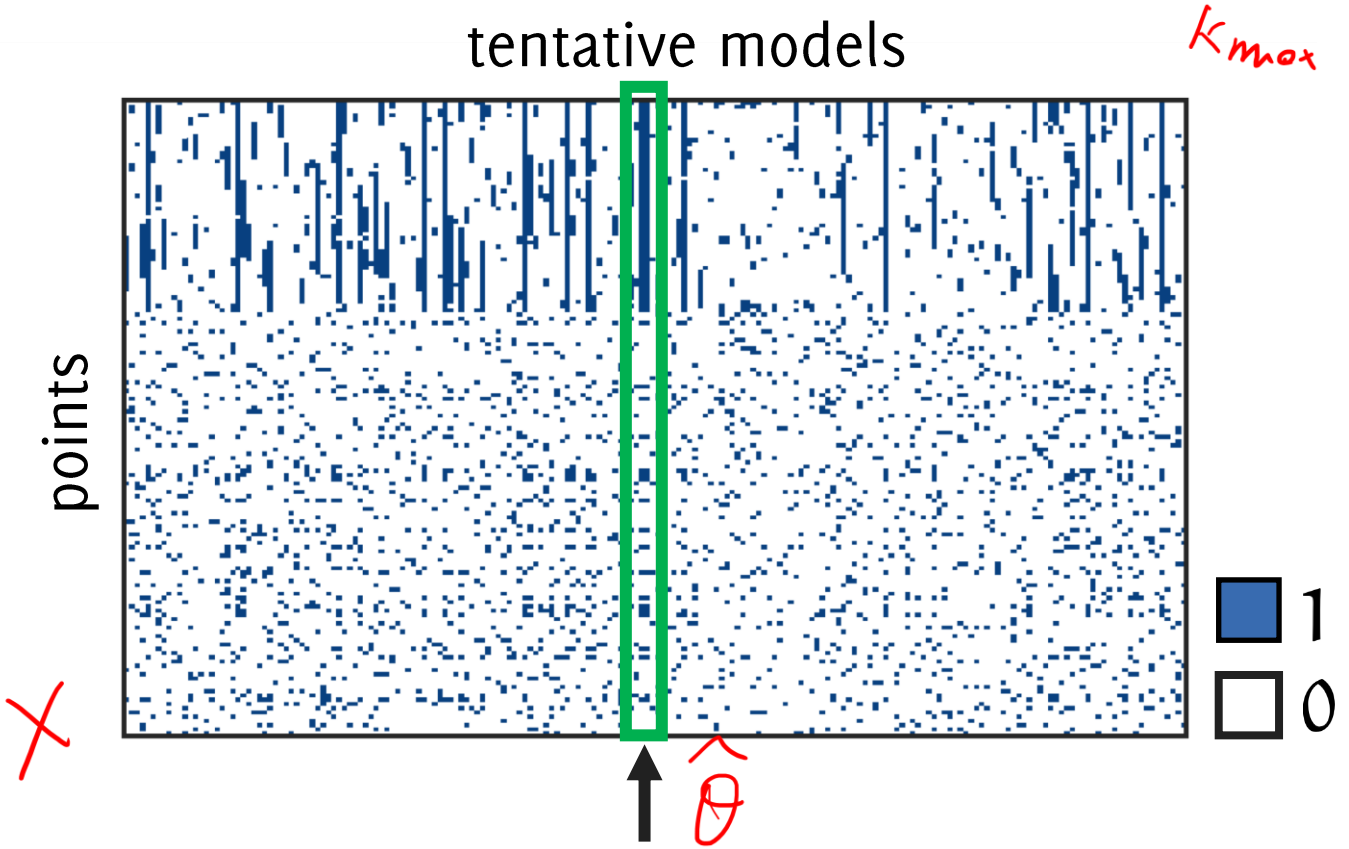


Data driven search of model space

$$H = \{\theta_1, \theta_2, \ldots, \theta_m\} \approx \Theta$$



pick the column with the maximum sum

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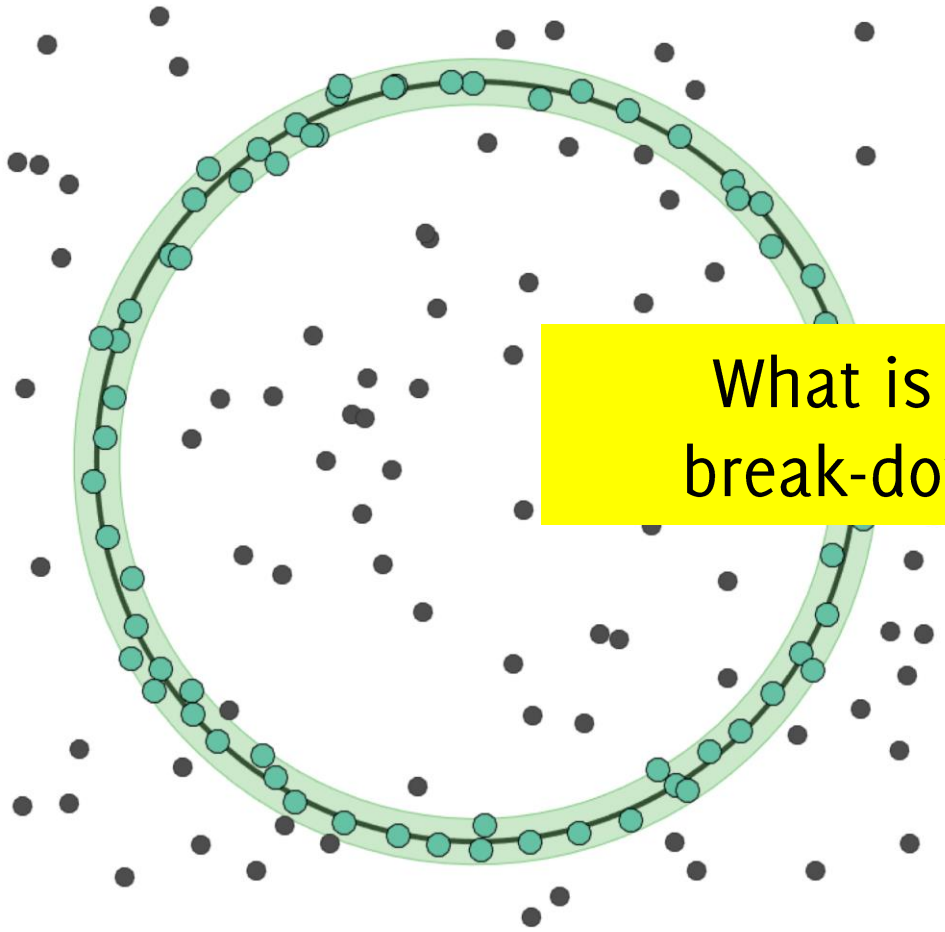
Randomized Sample Consensus



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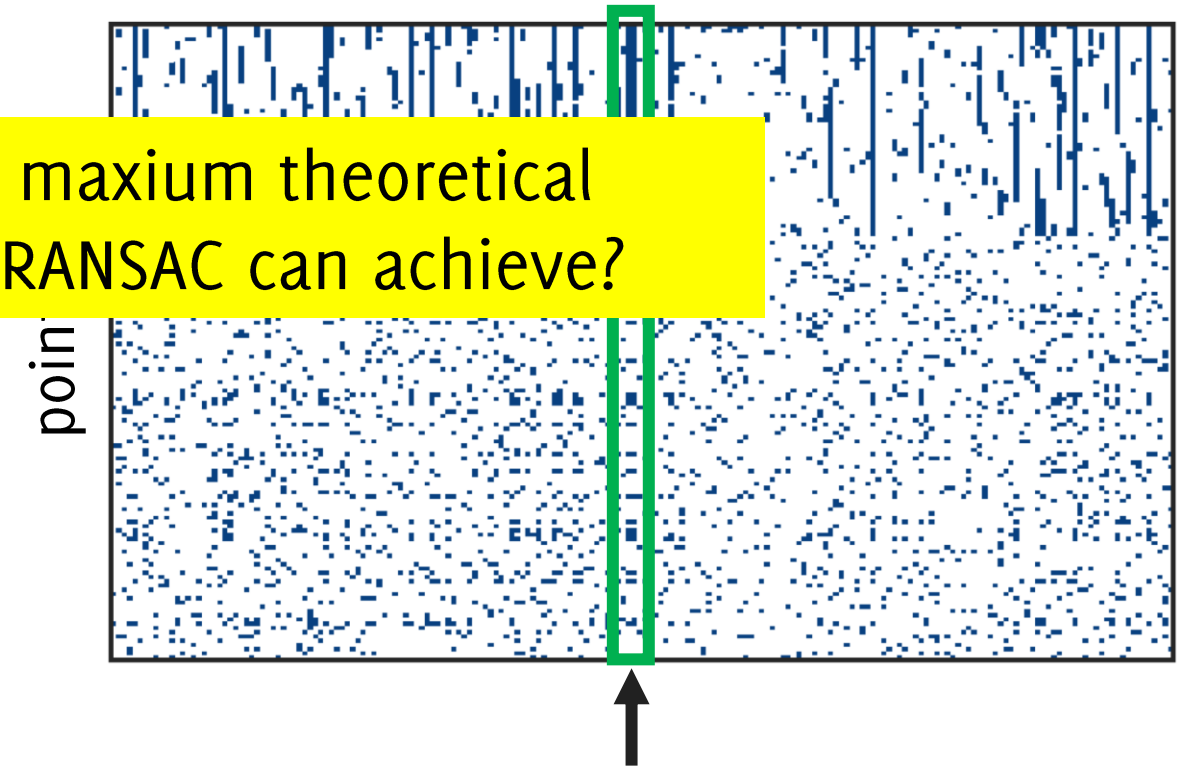
tentative models



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What is the maxium theoretical break-down RANSAC can achieve?

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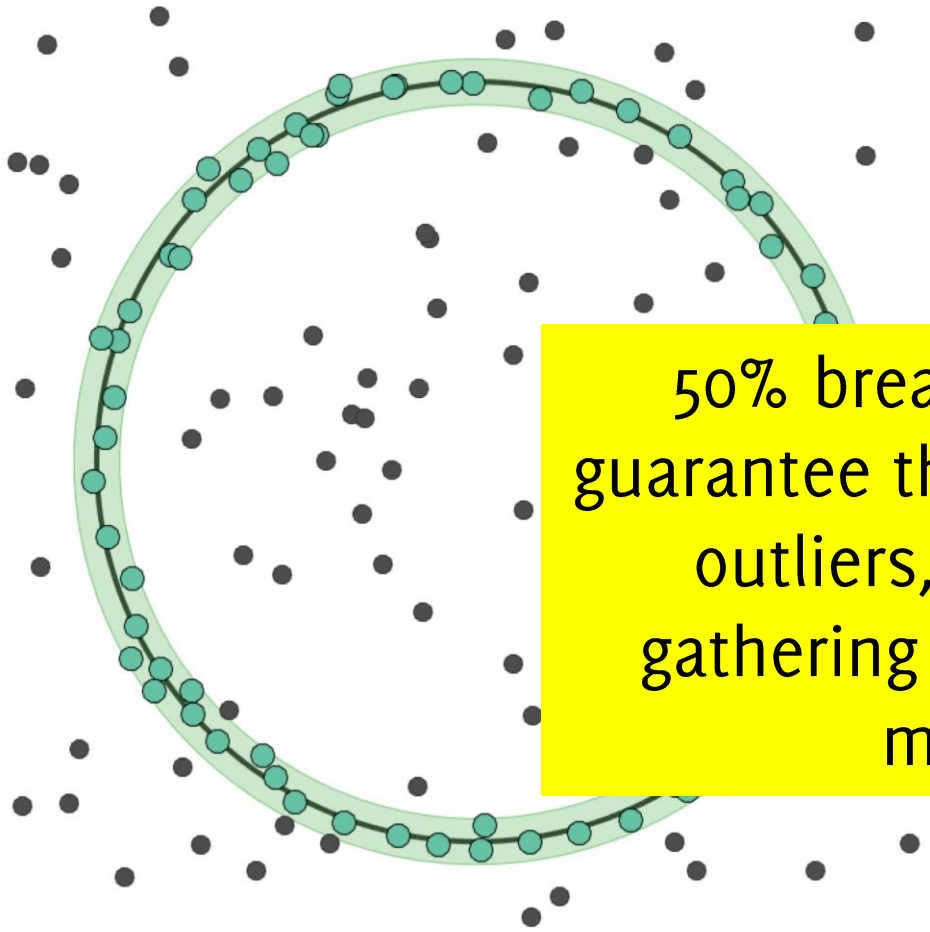
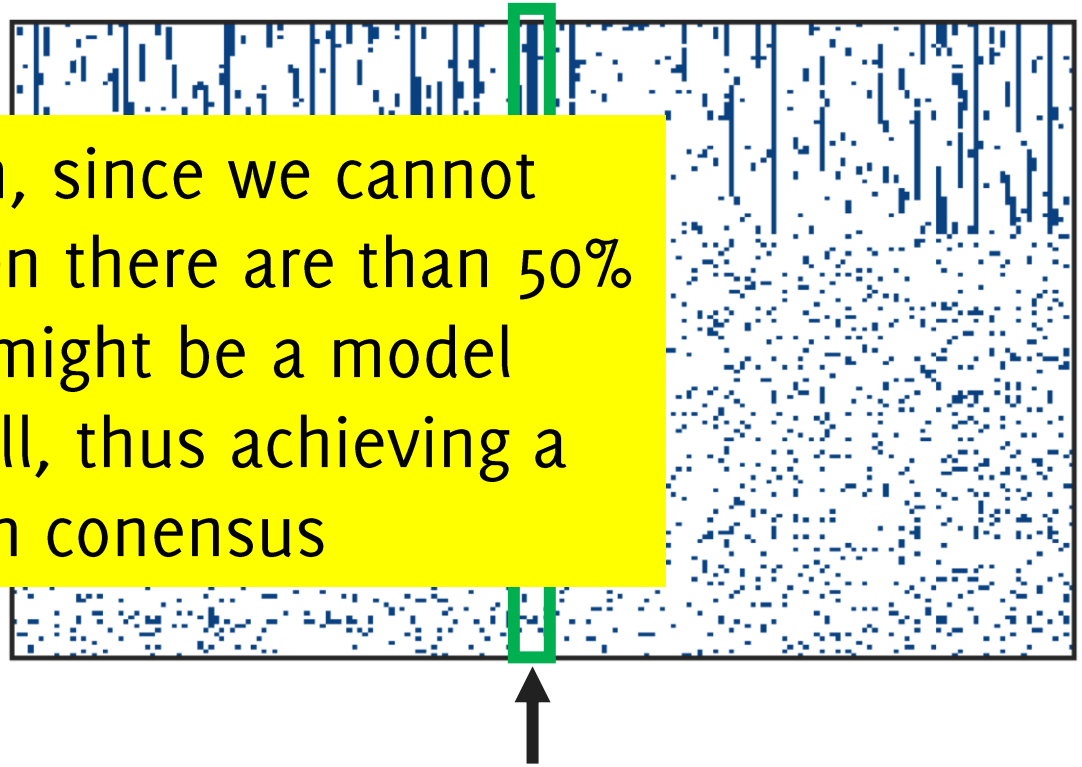
Randomized Sample Consensus



Data driven search of model space

$$H = \{\theta_1, \theta_2, \ldots, \theta_m\} \approx \Theta$$

tentative models



50% break-down, since we cannot guarantee that when there are than 50% outliers, there might be a model gathering them all, thus achieving a maximum conensus

1

0

pick the column with the maximum sum

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Ransac: practical issues

The size of the minimum sample S: this is the bare minimum number of points to fit the parametric model at hand

The inlier threshold ϵ: this can be estimated from the noise in the data

The number of iterations n (or k_{MAX}): the criteria for selecting the number of samples n: "Choose n so that, with probability p (e.g. $p = 0.99$), at least **one random sample is without outliers** "

The maximum number of iterations n

Let e the probability of a sample to be an outlier and $1 - e$ the probability of an inlier (can be estimated / provided by a-priori information)

The probability that all s points are inliers: $(1 - e)^s$

The probability that at least one point in S is an outlier: $1 - (1 - e)^s$ (this is the probability for a sample S to yield the right model)

The probability that all the n selected set contain outliers

$$(1 - (1 - e)^s)^n$$

The probability that at least one the n set is without outliers:

$$1 - (1 - (1 - e)^s)^n$$

Set n to have the above probability below a parameter p

$$p = (1 - (1 - (1 - e)^s)^n) \rightarrow n = \log(1 - p) / \log(1 - (1 - e)^s)$$

Ransac: practical issues

Choose n so that, with probability p, at least one random sample is free from outliers (e.g. $p = 0.99$) (outlier ratio: e)

$$n = \log(1-p) / \log(1-(1-e)^s)$$

	proportion of outliers e						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

Ransac: details

Repeat n times:

- Draw s points uniformly at random
- Fit line to these s points
- Find inliers to this line among the remaining points (i.e., points whose distance from the line is less than t)
- Update n
  - $e = 1 - \frac{\text{number of inliers}}{\text{number of points}}$
  - $n = \log(1-p) / \log(1-(1-e)^s)$

Choose the best model

Re-estimate the line with the inliers only through ordinary least square

Ransac

Pros:

- very popular (>22900 citations in Google Scholar)
- many improvements have been proposed
- very versatile
- agnostic on outlier percentage
- mild assumption: know the scale noise to set the inlier threshold ϵ

Cons:

- can take longer than expected

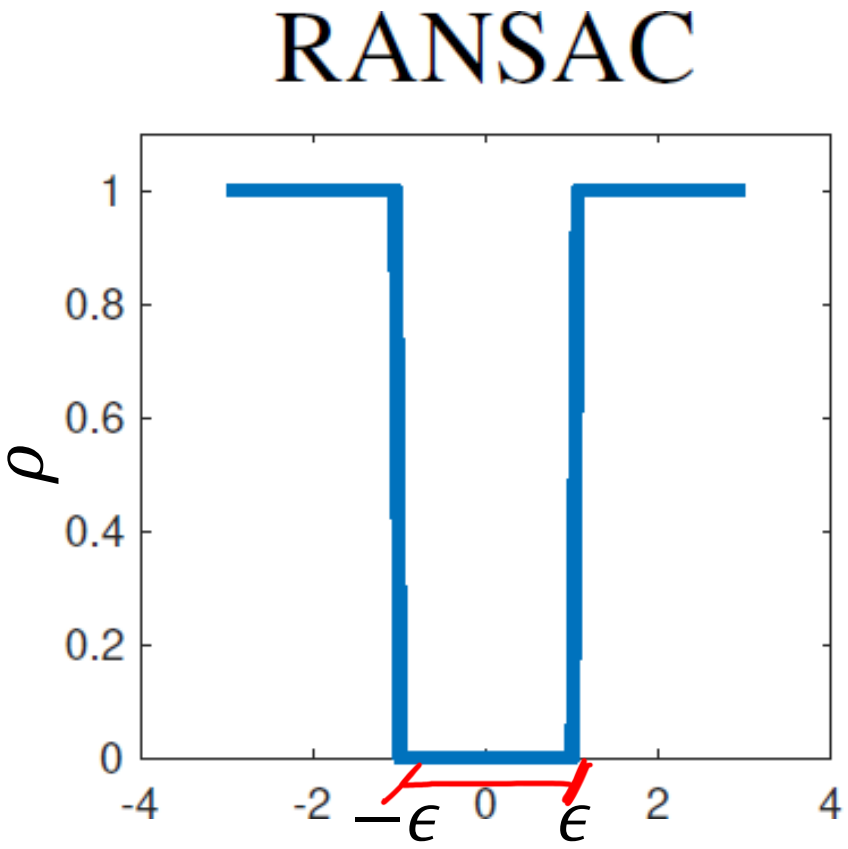
Ransac as M-estimator

(Steward 1999) RanSaC can be seen as a particular M-estimator since the **loss it minimizes** is the number of points having residual above the inlier threshold ϵ

$$f(r_i) = \begin{cases} 1, & r_i > \epsilon \\ 0, & r_i \leq \epsilon \end{cases}$$

Of course selecting inlier thrshold ϵ is very critical

Ransac achieves a theoretical breakdown of 50% of outliers, but in practice, provided a good selection of ϵ, this can be even higher



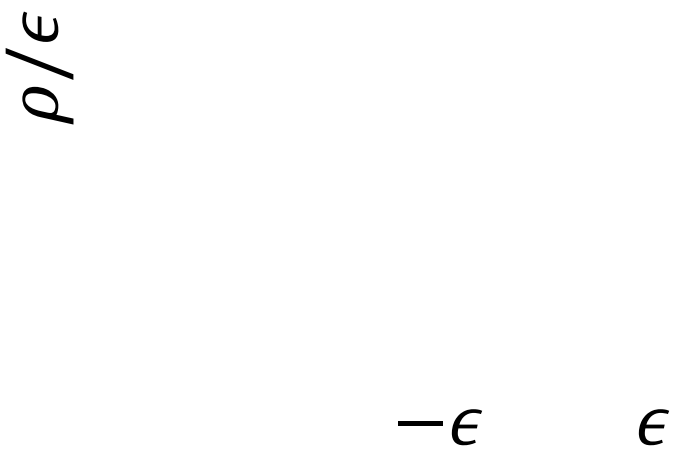
MSAC

(Torr and Zisserman 2000) a different loss function to be minimized within the RanSaC framework

$$f(r_i) = \begin{cases} \epsilon, & |r_i| > \epsilon \\ |r_i|, & |r_i| \leq \epsilon \end{cases}$$

This turns to be more effective and should be preferred to RanSaC

) a different loss function
ne RanSaC framework
$|r_i| > \epsilon$
$|r_i| \leq \epsilon$
ective and should be



Ransac vs MSaC

Input: X data, ϵ inlier threshold, k_{max} max iteration
Output: θ^* model estimate
$J^* = -\infty, k = 0;$
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \sum_{x \in X} \hat{f}_\epsilon(r(x, \theta))$;
  if $J(\theta) > J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

Input: X data, ϵ inlier threshold, k_{max} max iteration
Output: θ^* model estimate
$J^* = +\infty, k = 0;$
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Estimate inlier set $I = \{x \in X : r(x, \theta)^2 < \epsilon^2\}$;
  Evaluate $J(\theta) = \sum_{x \in I} r(x, \theta) + (|X| - |I|)\epsilon$;
  if $J(\theta) < J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

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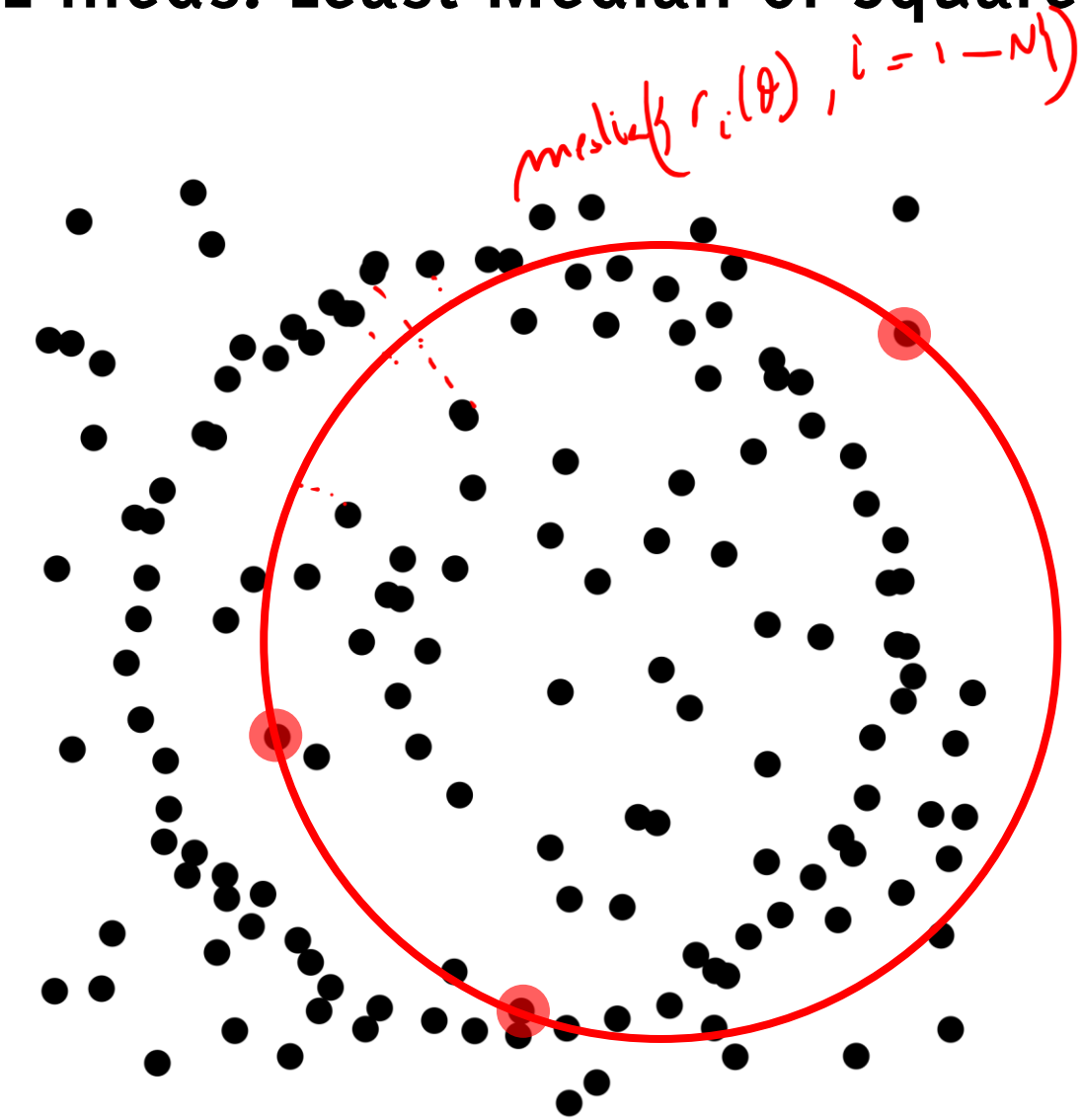
Least Median of Squares

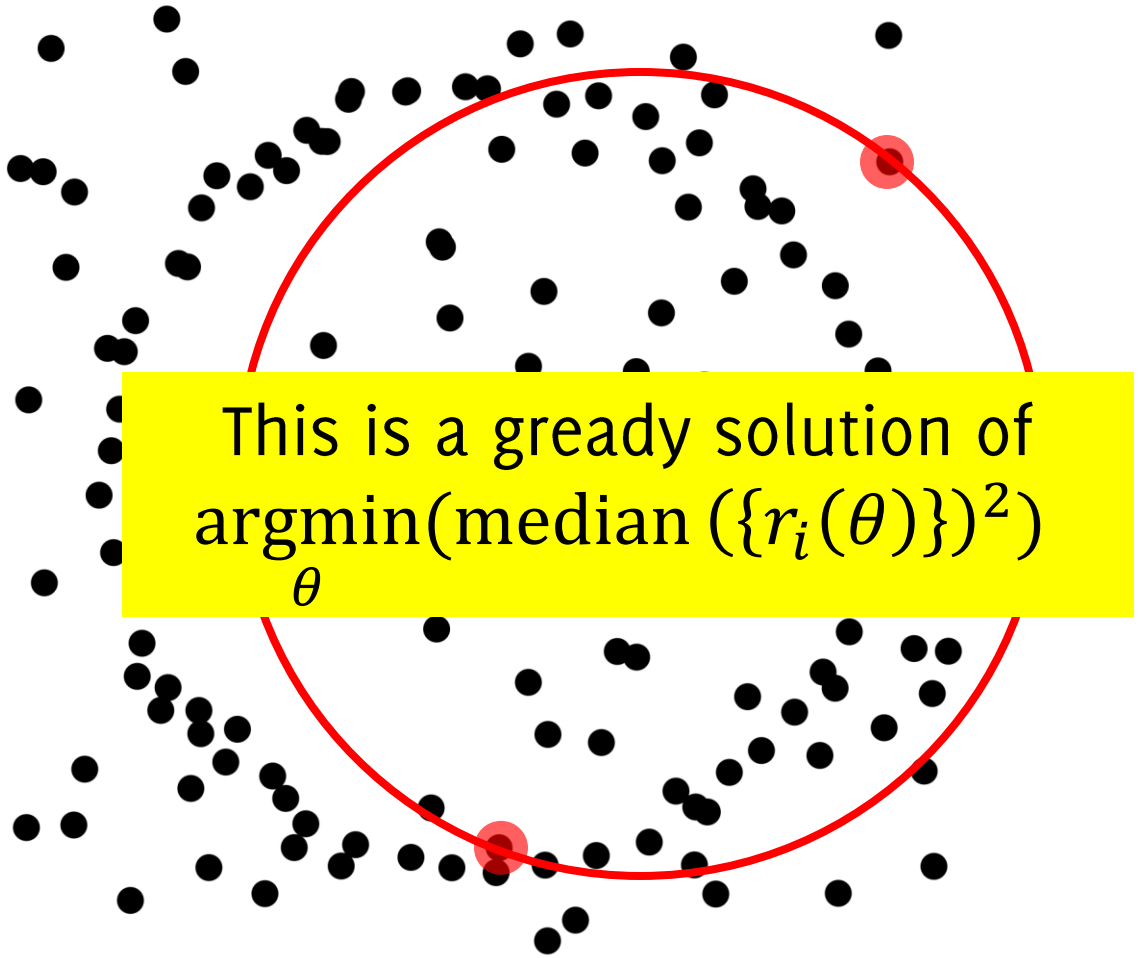
L-meds: Least Median of Squares, Rousseeuw e Leroy (1987)



Input: X data, k_{max} max iteration
Output: θ^* model estimate
$J^* = +\infty$, $k = 0$;
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \text{median}_{x \in X}(r(x, \theta))$;
  if $J(\theta) < J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

Credits Luca Magri,

L-meds: Least Median of Squares, Rousseeuw e Leroy (1987)



This is a gready solution of $\underset{\theta}{\operatorname{argmin}}(\operatorname{median}(\{r_i(\theta)\})^2)$

Input: X data, k_{max} max iteration
Output: θ^* model estimate
$J^* = +\infty, k = 0;$
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \text{median}_{x \in X}(r(x, \theta))$;
  if $J(\theta) < J^*$ **then**
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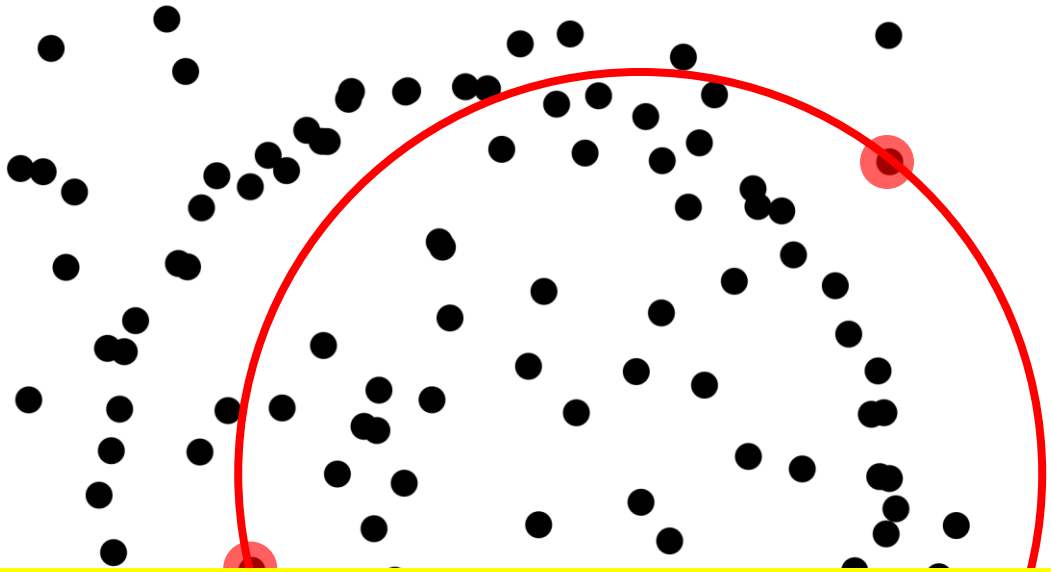
Optimize θ^* on its inliers.

Credits Luca Magri,

L-meds: Least Median of Squares, Rousseeuw e Leroy (1987)

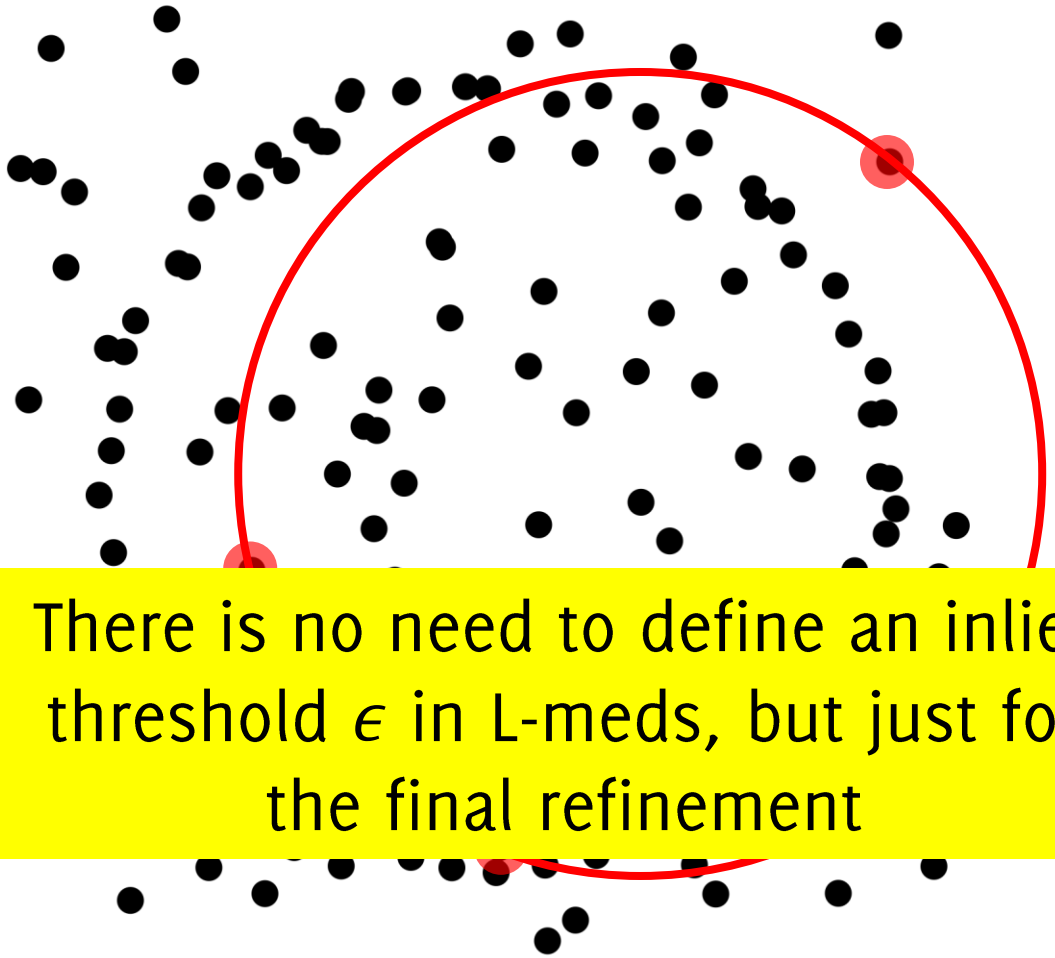


Since there is no explicit definition of inliers here, inliers can be identified as points having residuals (w.r.t. to the final model) that are smaller than 2.5σ

Input: X data, k_{max} max iteration
Output: θ^* model estimate
$J^* = +\infty, k = 0;$
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \text{median}_{x \in X}(r(x, \theta))$;
  if $J(\theta) < J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

Credits Luca Magri,

L-meds: Least Median of Squares, Rousseeuw e Leroy (1987)



There is no need to define an inlier threshold ϵ in L-meds, but just for the final refinement

Input: X data, k_{max} max iteration
Output: θ^* model estimate
$J^* = +\infty$, $k = 0$;
repeat
  Select randomly a minimal sample set $S \subset X$;
  Estimate parameters θ on S;
  Evaluate $J(\theta) = \text{median}_{x \in X}(r(x, \theta))$;
  if $J(\theta) < J^*$ **then**
    $\theta^* = \theta$;
    $J^* = J(\theta)$;
  end
  $k = k + 1$;
until $k > k_{max}$;
Optimize θ^* on its inliers.

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Assigments

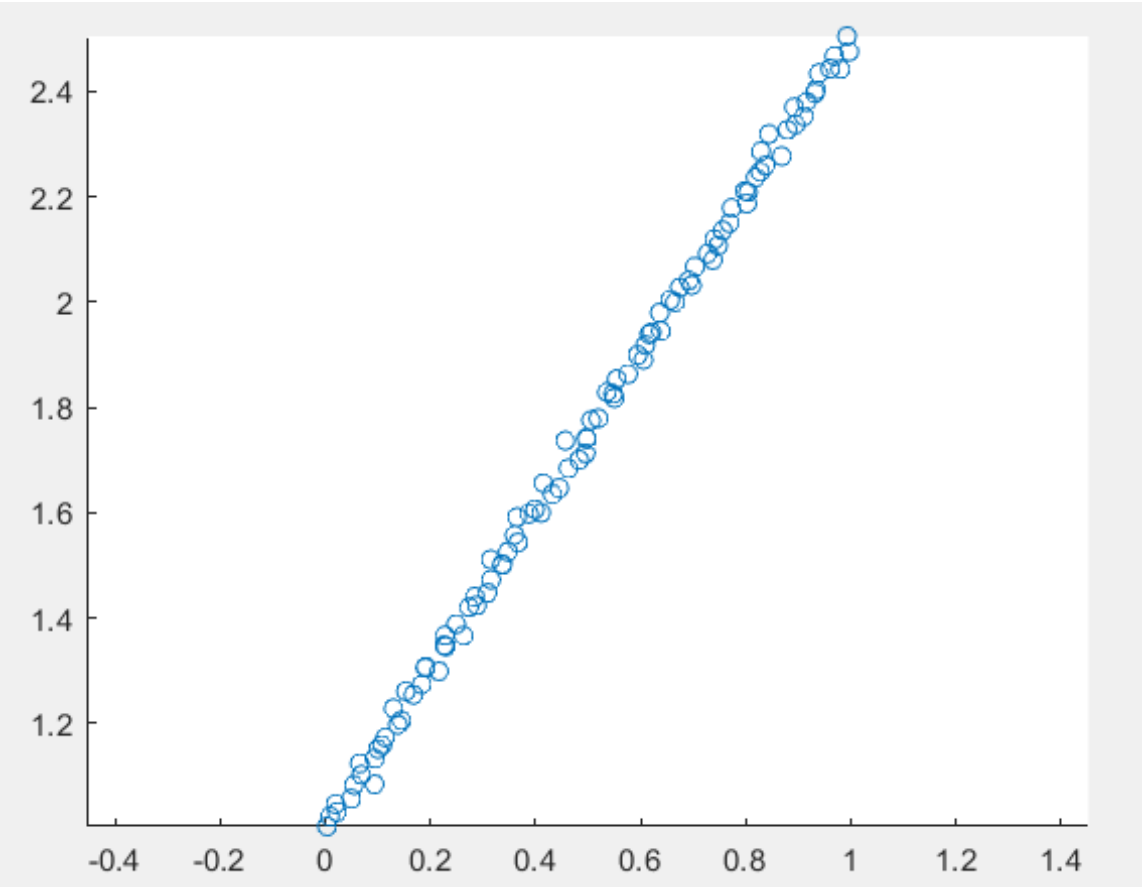
Fitting Functions

Implement

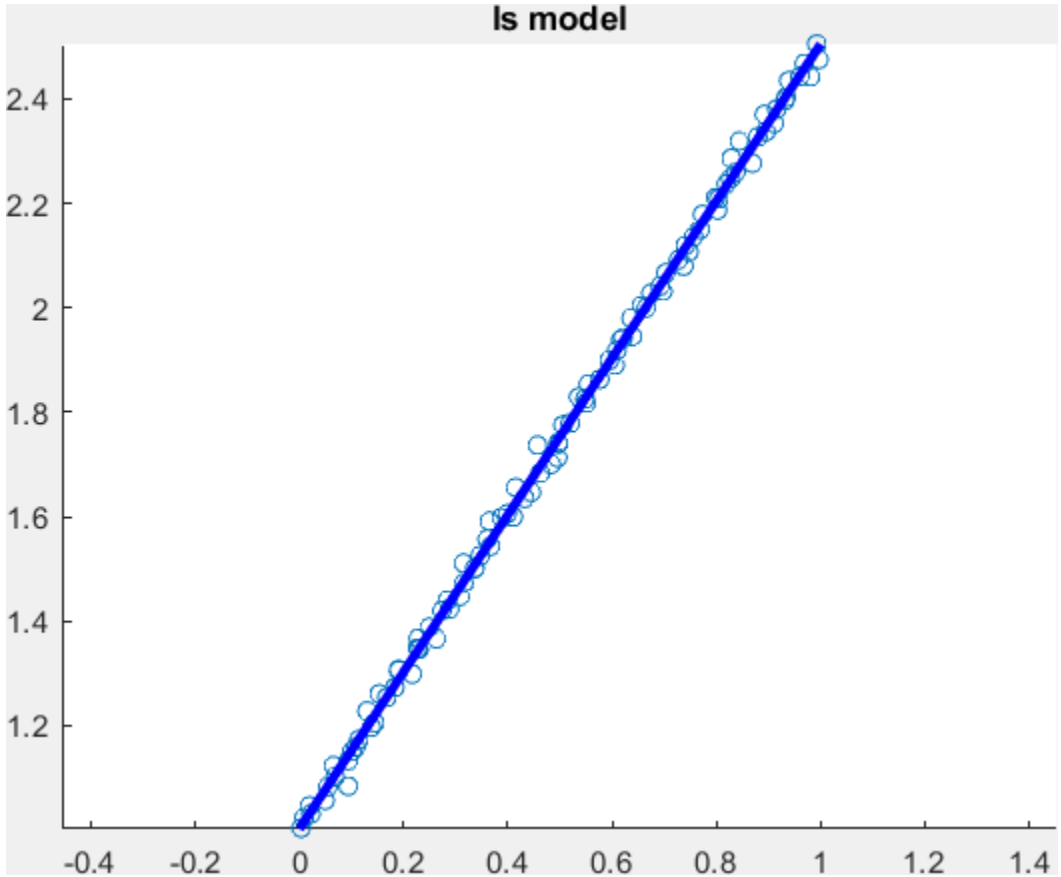
- **`fit_line_ols`** to perform line fitting using Ordinary Least Square (minimization of algebraic error, corresponding to vertical distances)
- **`fit_line_DLT`** to perform line fitting minimizing the geometric distance

Invoke these from **`demo_robustmf_TODO`** and **`test robustness to outliers`**

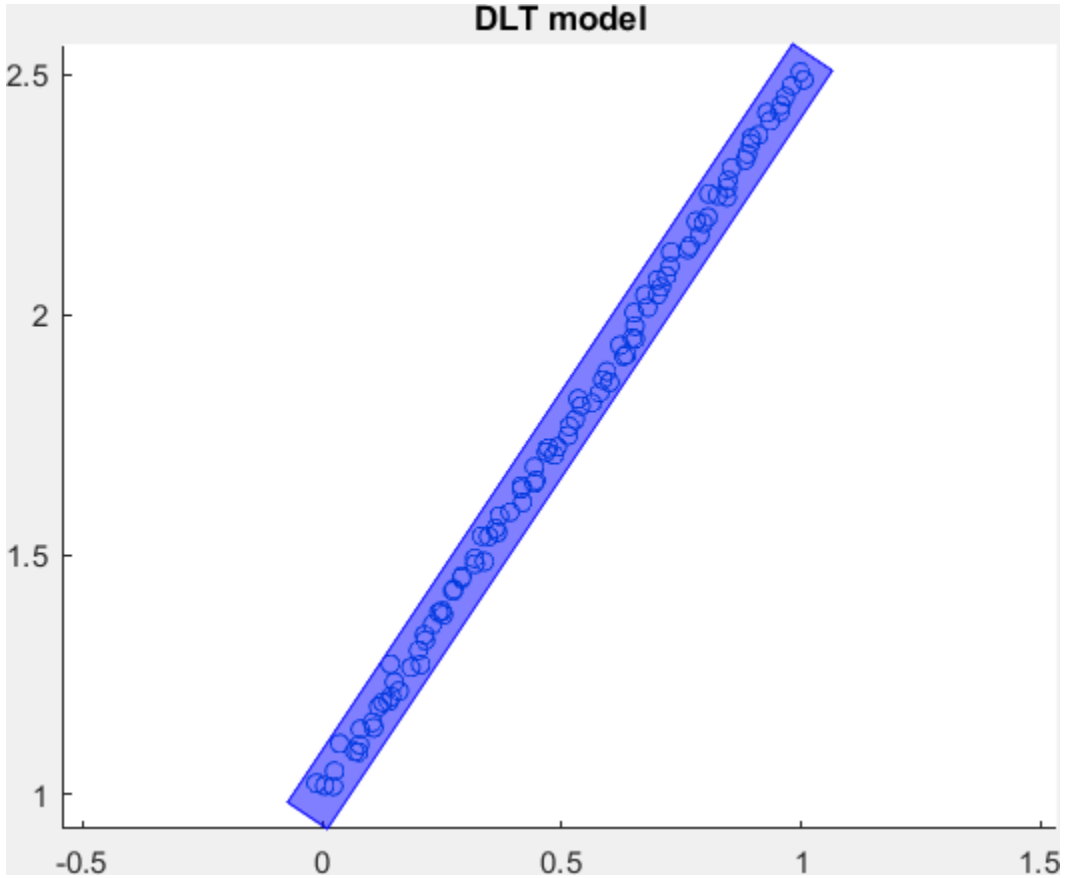
Here are the data



Least square fit

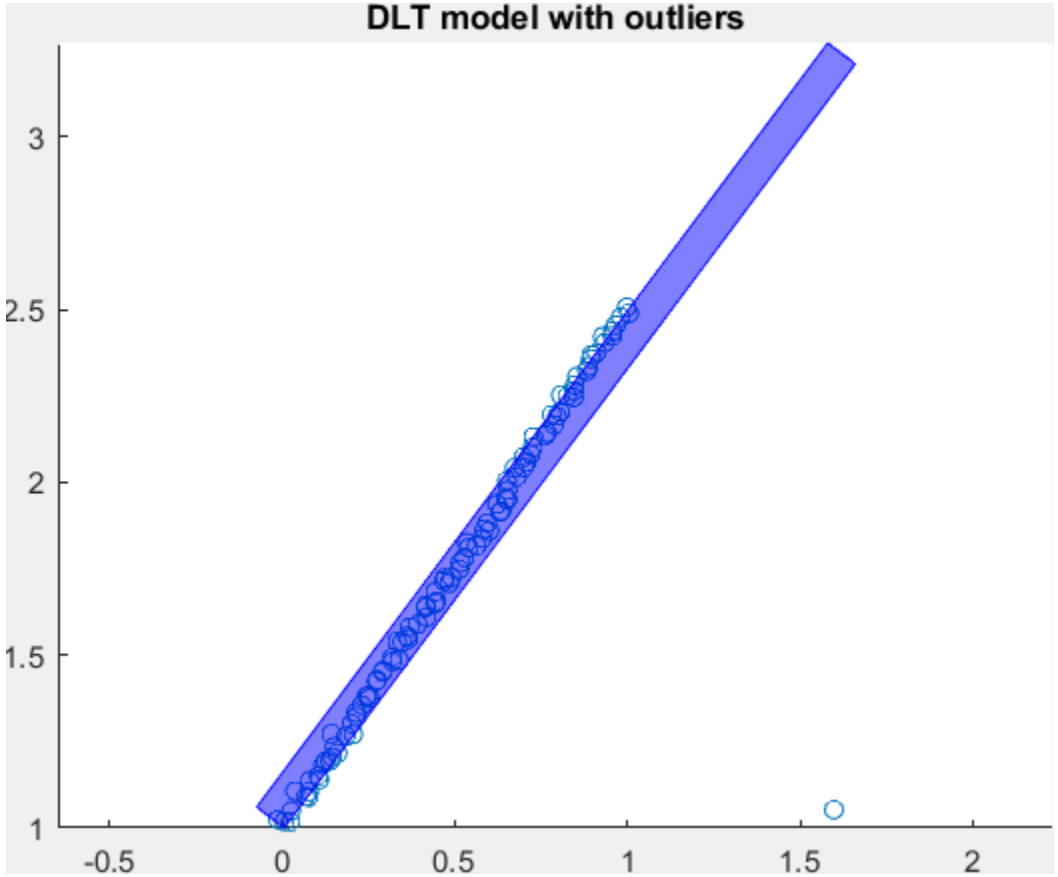


Inlier Bands



G. Boracchi

A single outlier can impact the fitting



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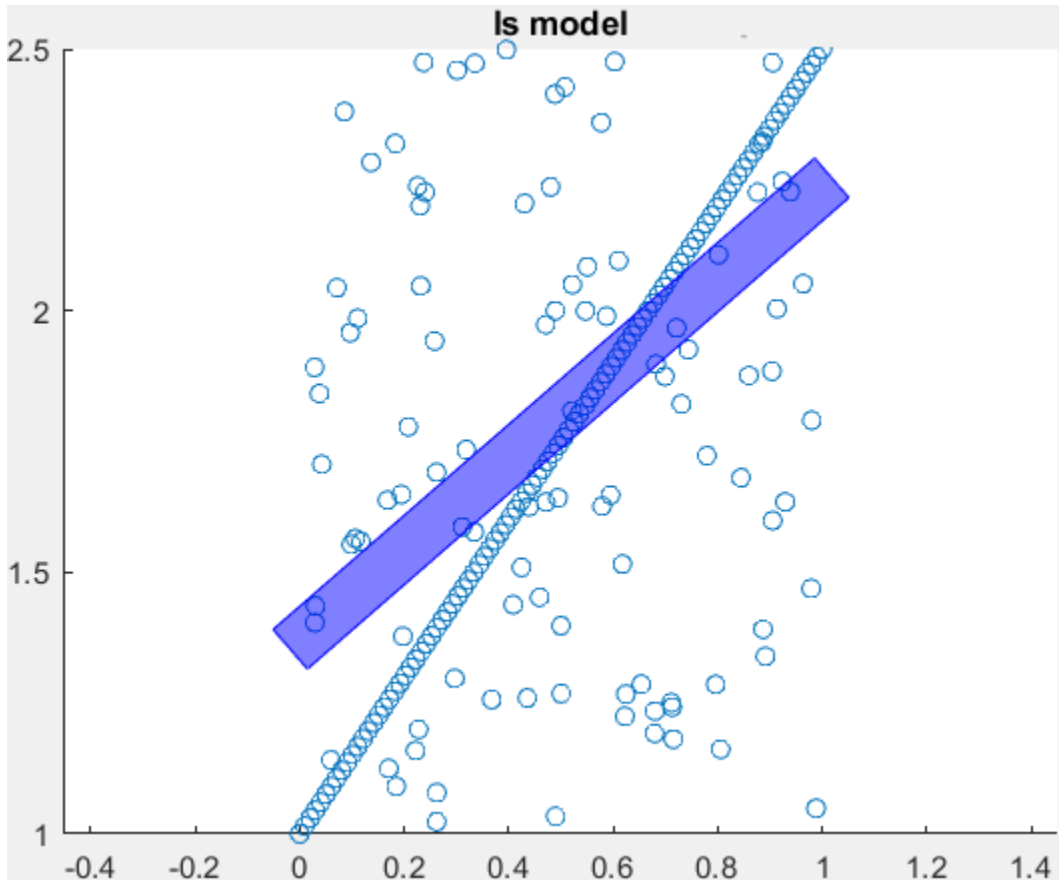
50% outliers, 50% inliers



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