Course Introduction and Logistics

Mathematical Models and Methods for Image Processing

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https://boracchi.faculty.polimi.it/

February 18th 2025

The Team

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Mathematician (Università Statale degli Studi di Milano 2004),

PhD in Information Technology (DEIB, Politecnico di Milano 2008)

Associate Professor since 2019 at DEIB (Computer Science), Polimi

Research Interests are mathematical and statistical methods for:

- Image / Signal analysis and processing
- Unsupervised learning, change / anomaly detection

Edoardo Peretti

Computer Science and Mathematical Engineering (Polimi 2022),

Currently PhD student in Information Technology (since 2023)

Research Interests are mainly focused on:

- Mathematical models for Image / Video Denoising
- Deep Learning for Image Restoration
- Change Detection in DataStreams

The Course

The Goal

The primary ***goal*** *of this laboratory course is to* ***let the students design, implement and practice algorithms*** *based* ***on***

- ***simple mathematical models*** *from* ***linear algebra and convex optimization,***
- ***solve*** *challenging inverse* ***problems in image processing*** *(denoising, deblurring, inpainting, anomaly detection)*
- *Understand the most important aspects of* ***sparse representations*** *and of sparsity as a form of* ***regularization in learning problems****.*

What you get

- *An **excellent opportunity to practice** and gain better insights on fundamental principles and techniques (linear algebra, convex optimization)*

- *Gain the **fundamental notions** and expertise to approach many **image processing problems** and take advantage your mathematical background*

The outline

The course **topics include**:

- **Image models based on orthonormal bases** (Fourier, wavelets), **data-driven basis** (PCA, Gram-Schmidt) and **local polynomial approximation**.
- **Sparsity and redundancy**.
  - Away from Orthonormal Basis, redundant set of generators
  - Sparse coding with ℓ^0 (OMP) or ℓ^1 norm (convex optimization ISTA, IRLS, LASSO)
  - Dictionaries yielding sparse representations and dictionary learning (KSVD)
- **Applications of sparse models** to image denoising, inpainting, anomaly detection and classification.

- **Robust fitting** methods (RANSAC, LMEDS, HOUGH) and their sequential counterparts for object detection in images.

The Materials

- Very few slides, lectures at the backboard!
  - Yes, you need to take notes…
- Code snippets to be filled in will be provided
  - Both Python and Matlab are accepted. Consider I am a native Matlab speaker, Diego is more Python oriented.
- Please refer to the website

The Lectures

The Lectures

There is no a dramatic difference between lectures and laboratory

Most often, in both cases there will be:

- Some recap on background notions
- Something new: an algorithm, a method, the solution for a specific application
- Some guided practical session

All the materials can be found on the course website:
https://boracchi.faculty.polimi.it/teaching/MMMIP.htm

The Exam

The exam

The exam consists in:

- Solving the **assignment** provided during lectures (to be delivered by23.59 GMT of the exam day).
- An **oral exam** about the course materials (schedule to be defined in 10-15 days after submitting homeworks).

The assignments are given during lectures with the purpose of:

- Let you familiarize and put in practice the presented models and methods
- Make sure you understood the algorithms

Assessment Criteria:

- Ability to illustrate algorithms and theory behind them
- Understanding of models and their use in applications
- Active participation during laboratories and laboratories

Frequently Asked Questions

Q: Any specific background?

A: linear algebra, statistics and calculus

Q: Any programming skill required?

A: Proficiency in Matlab or Python

Q: Plenty of neural networks then?

*A: No way. No neural networks allowed here** ☺*
Only expert-driven algorithms designed upon a clear mathematical modeling that admits closed-form solutions / sound optimization schemes.

* Interested in neural networks? Refer to «Artificial Neural Network and Deep Learning» in the first semester

Questions?

This is the fourth edition of the course… but we are always open to changes and improvements.

We might need to adjust quite a few things …. your feedback in this regard is very precious!

Denoising over adaptively defined neighborhoods for local polynomial regression

Sparsity

From “**Sparse Modeling for Image and Vision Processing**”

J. Mairal, F.Bach, J.Ponce

Now Publisher 2012

Sparsity and Parsimony

The principle of sparsity or “parsimony” consists in *representing some phenomenon with as few variable as possible*

Stretch back to philosopher William Ockham in 14th Century

Wrinch and Jeffreys [1921] relate simplicity to parsimony:

> *The **existence of simple laws** is, then, apparently, to be regarded as **a quality of nature**; and accordingly we may infer that it is justifiable to **prefer a simple law to a more complex one that fits our observations slightly better**.*

Simplicity <-> number of learning parameters

Sparsity in Statistics

Statistics: simple models are preferred.

Sparsity is used to **prevent overfitting and improve interpretability of learned models.**

In model fitting, the number of parameters is typically used as a criterion to perform model selection.

See Bayes Information Criterion (BIC), Akaike Information Criterion (AIC), …., Lasso.

Sparsity in Signal Processing

Signal Processing: similar concepts but different terminology. **Vectors corresponds to signals** and **data modeling is crucial** for performing various operations such as **restoration, compression, solving inverse problems**.

Signals are approximated by sparse linear combinations of **prototypes** (basis elements / atoms of a dictionary), resulting in simpler and compact model.

Best subset selection <-> computing the sparse representation of a signal w.r.t. a give basis/dictionary

Neuroscience: Olshausen and Field [1996], learning the from a training set of data dictionaries yielding sparse representations

First Assignment

Today's Assignment: Generate the Basis

- Generate the DCT basis according to the following formula (DCT type II) the k-th atom of the DCT basis in dimension M is defined as

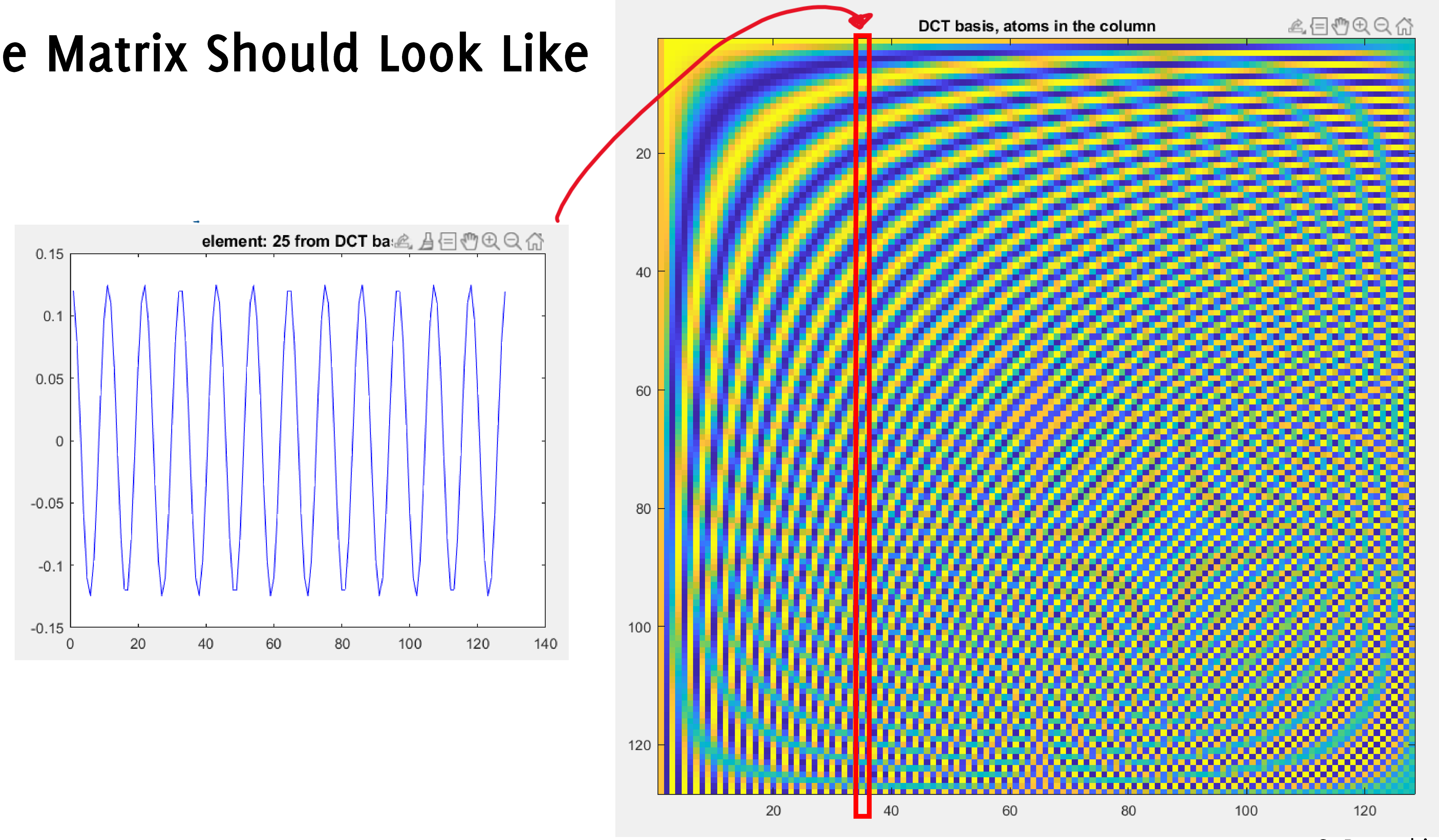
$$DCT_k(n) = c_k \cos\left(k\pi \frac{2n+1}{2M}\right) \quad n, k = 0, \dots, M-1$$

where $c_0 = \sqrt{1/M}$ and $c_k = \sqrt{2/M}$ for $k \neq 0$.

- For each $k = 0, \dots, M-1$, just sample each function $\cos\left(k\pi \frac{2n+1}{2M}\right)$ at $n = 0, \dots, M-1$, obtain a vector. Ignore the normalization coefficient. Divide each vector by its ℓ_2 norm.

- How can you use the function dct and it's inverse idct to define the DCT matrix?

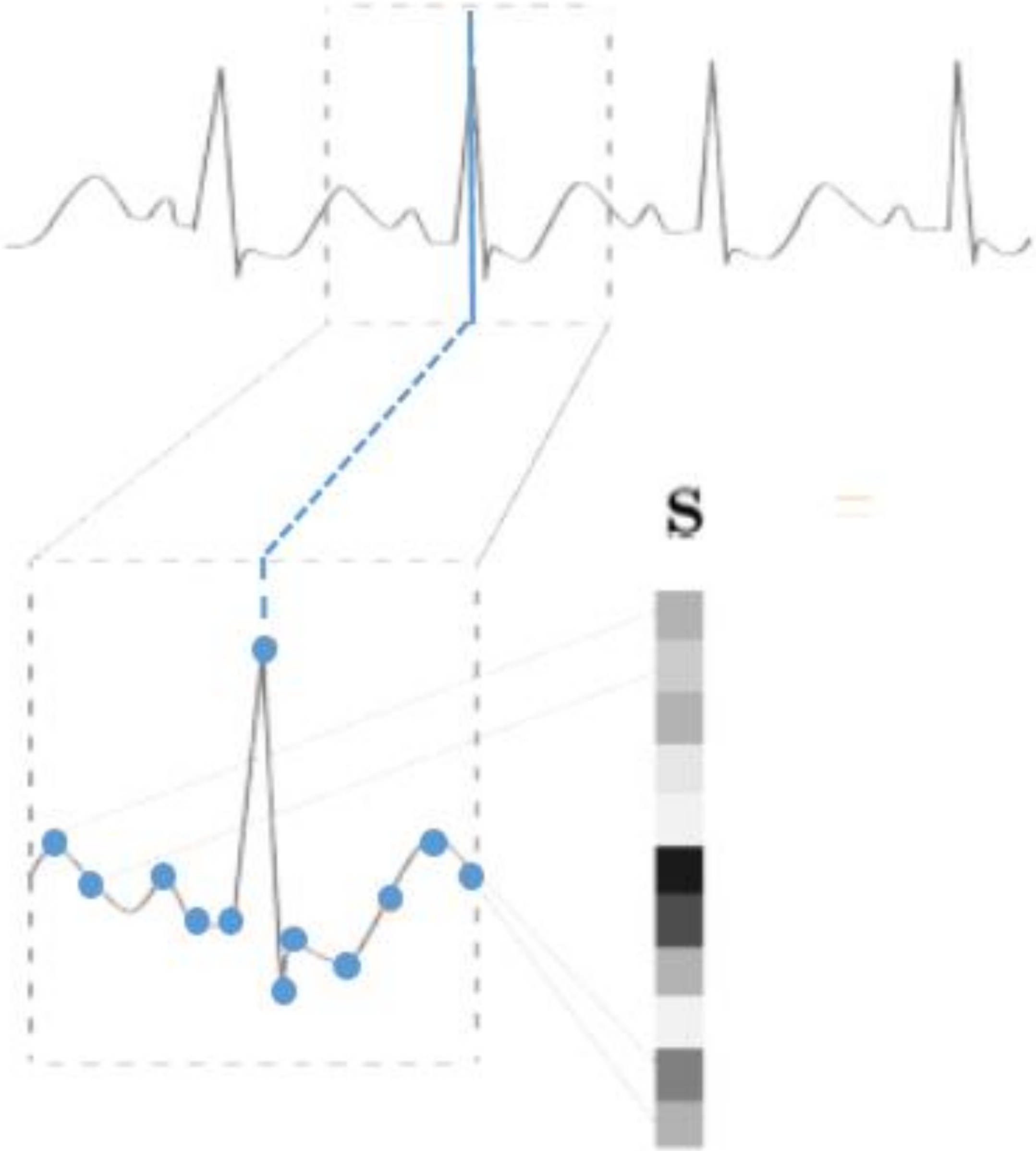
The Matrix Should Look Like



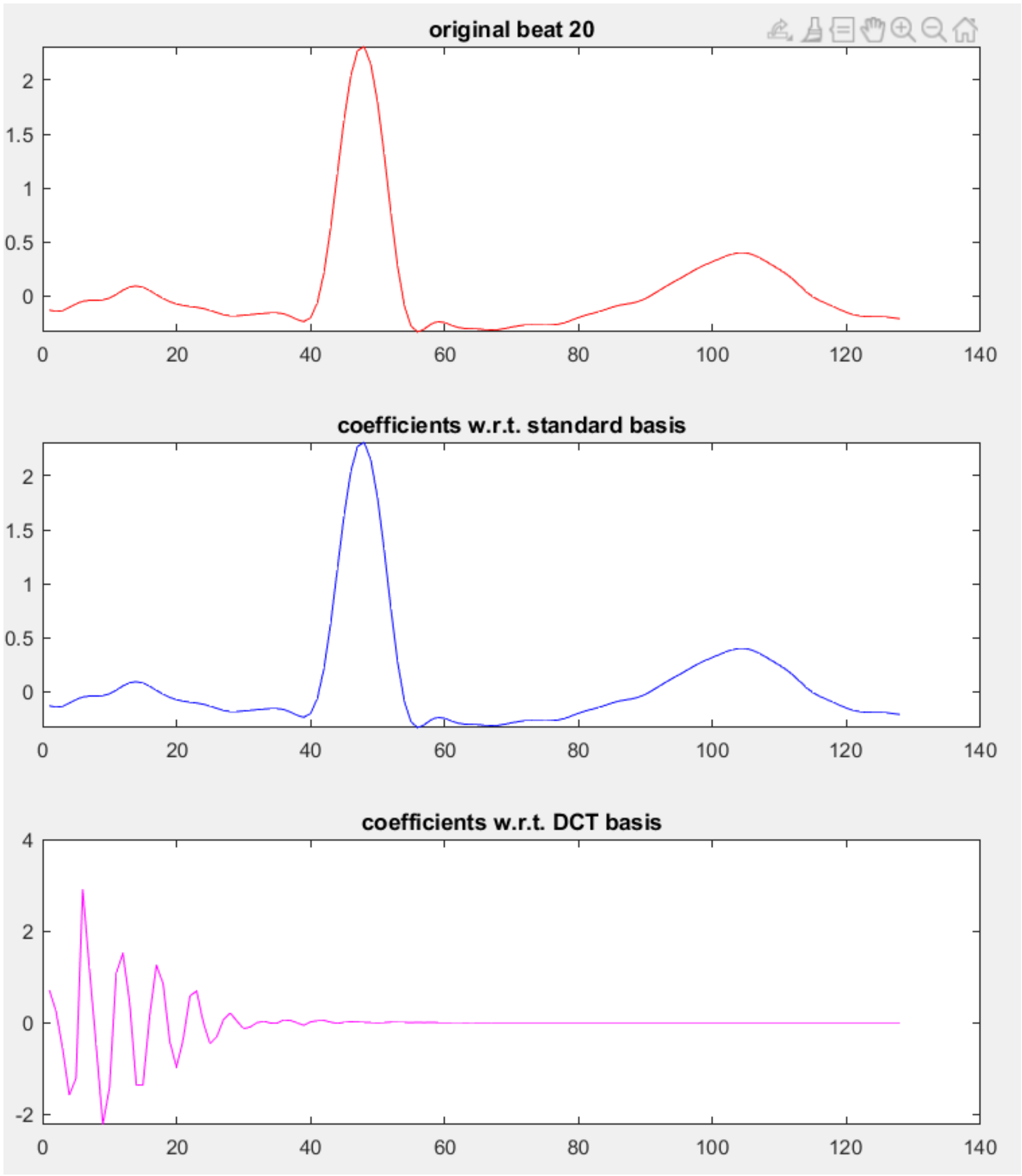
Today's Assignment: Analysis and Synthesis

- Load the ECG traces
- Analysis: Use the DCT basis you have defined to compute the representation of each signal $\boldsymbol{s}$ w.r.t the basis
- Display the coefficients and check whether they are sparse
- Synthesis: Reconstruct the signal from the coefficients. Check whether the reconstruction is perfect
- Add noise

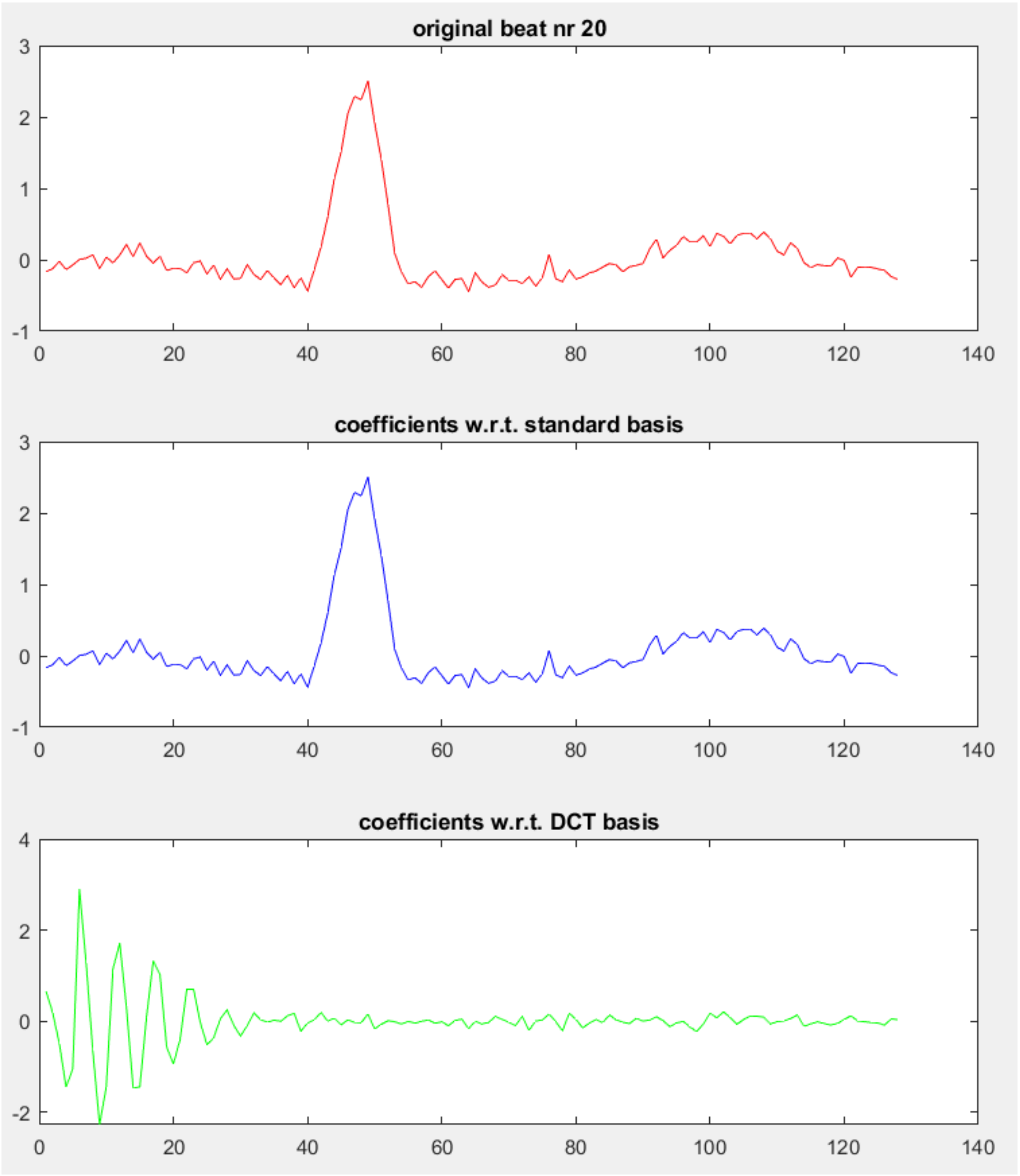
Modeling Scheme



w/o noise



w/ noise



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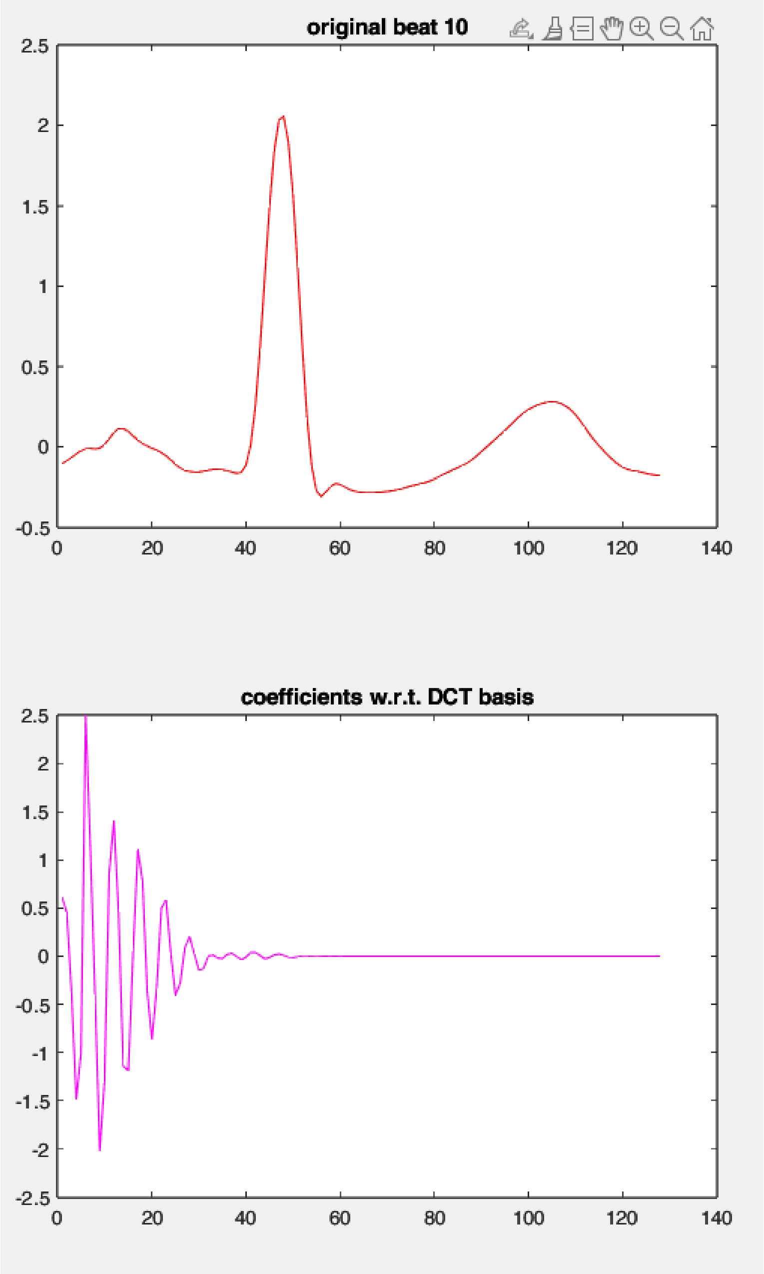
w/o noise

w/ noise



The sparsity prior seems to be effective on this type of signals.
There are **only few nonzero coefficients** in here.

DCT bases yields a **sparse representation** of heartbeats

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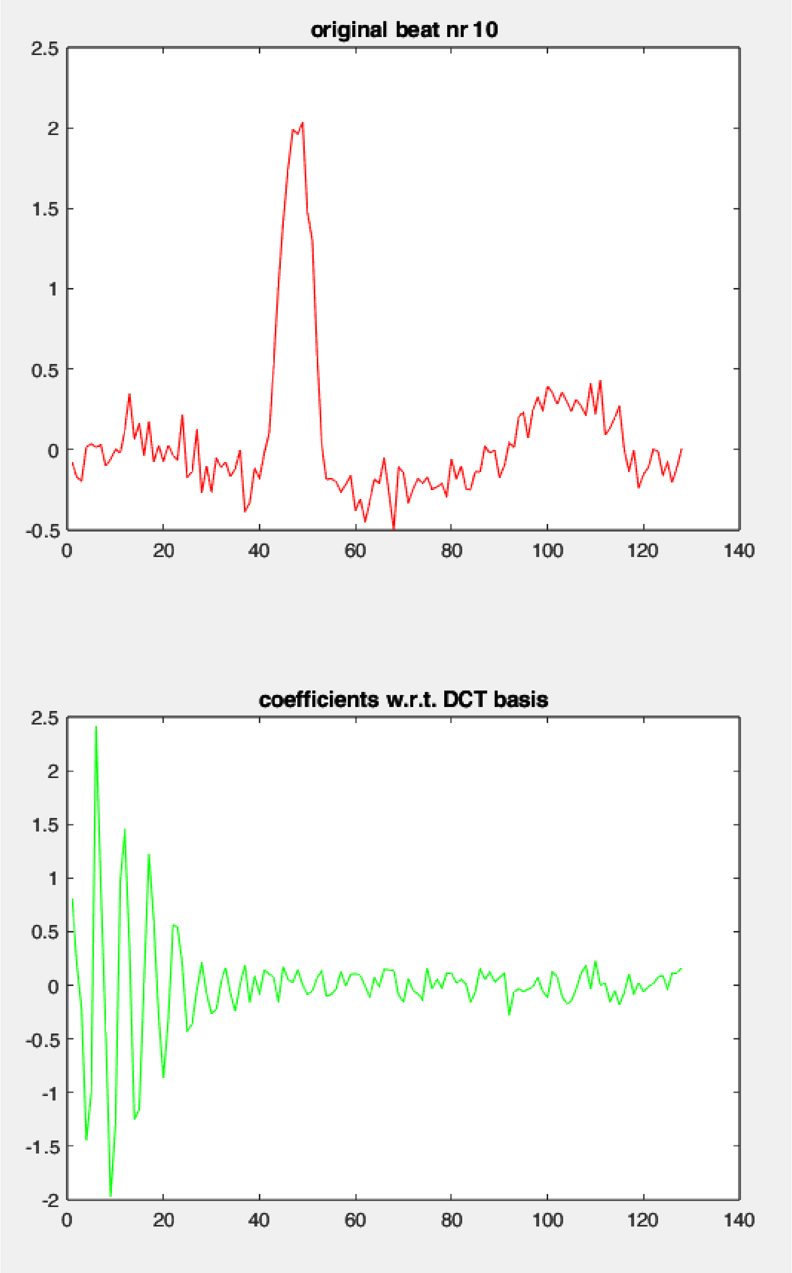
w/o noise

w/ noise

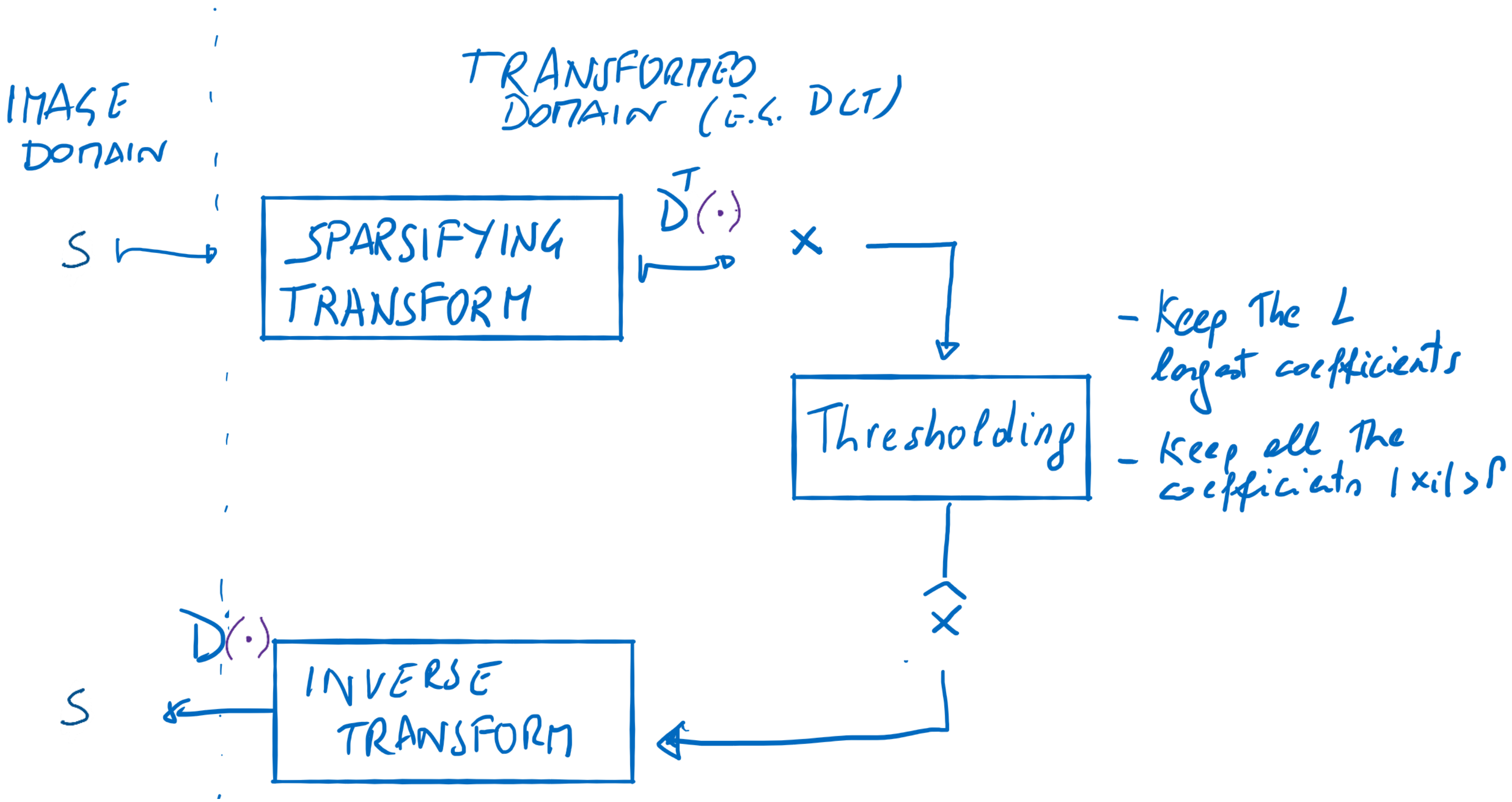


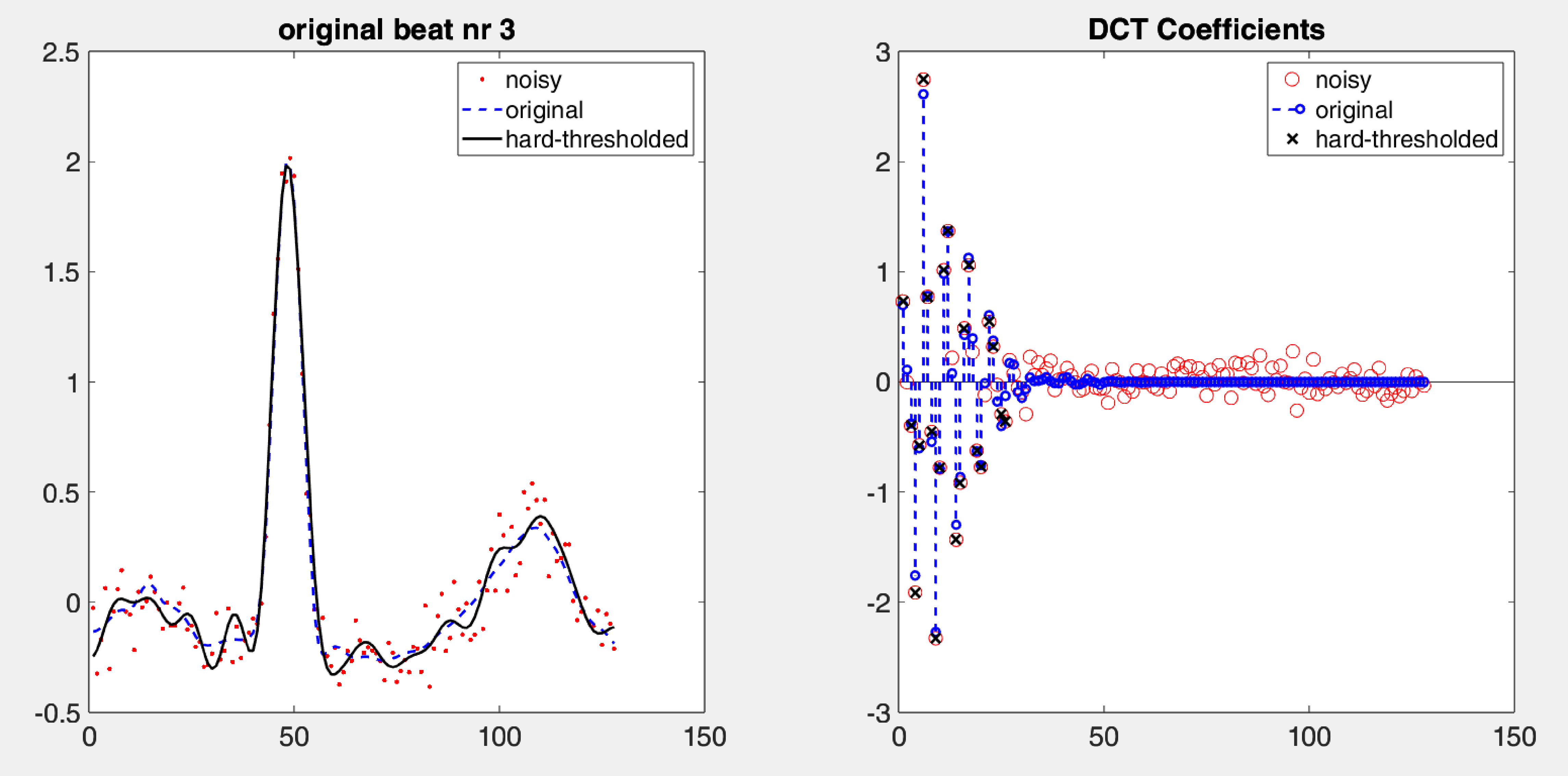
Still, the **coefficients referring to the noise-free** signal have a **magnitude** that is **larger** than those coefficients that are only affected by noise

However, the **large coefficients are also affected by noise** (therefore getting rid of the smallest coefficients won't return the noise-free signal)

Today's Assignment: Enforce Sparsity

- Enforce Sparsity to get rid of noise




original beat nr 3
noisy
original
hard-thresholded
2.5
2
1.5
1
0.5
0
-0.5
0
50
100
150
DCT Coefficients
noisy
original
hard-thresholded
3
2
1
0
-1
-2
-3
0
50
100
150

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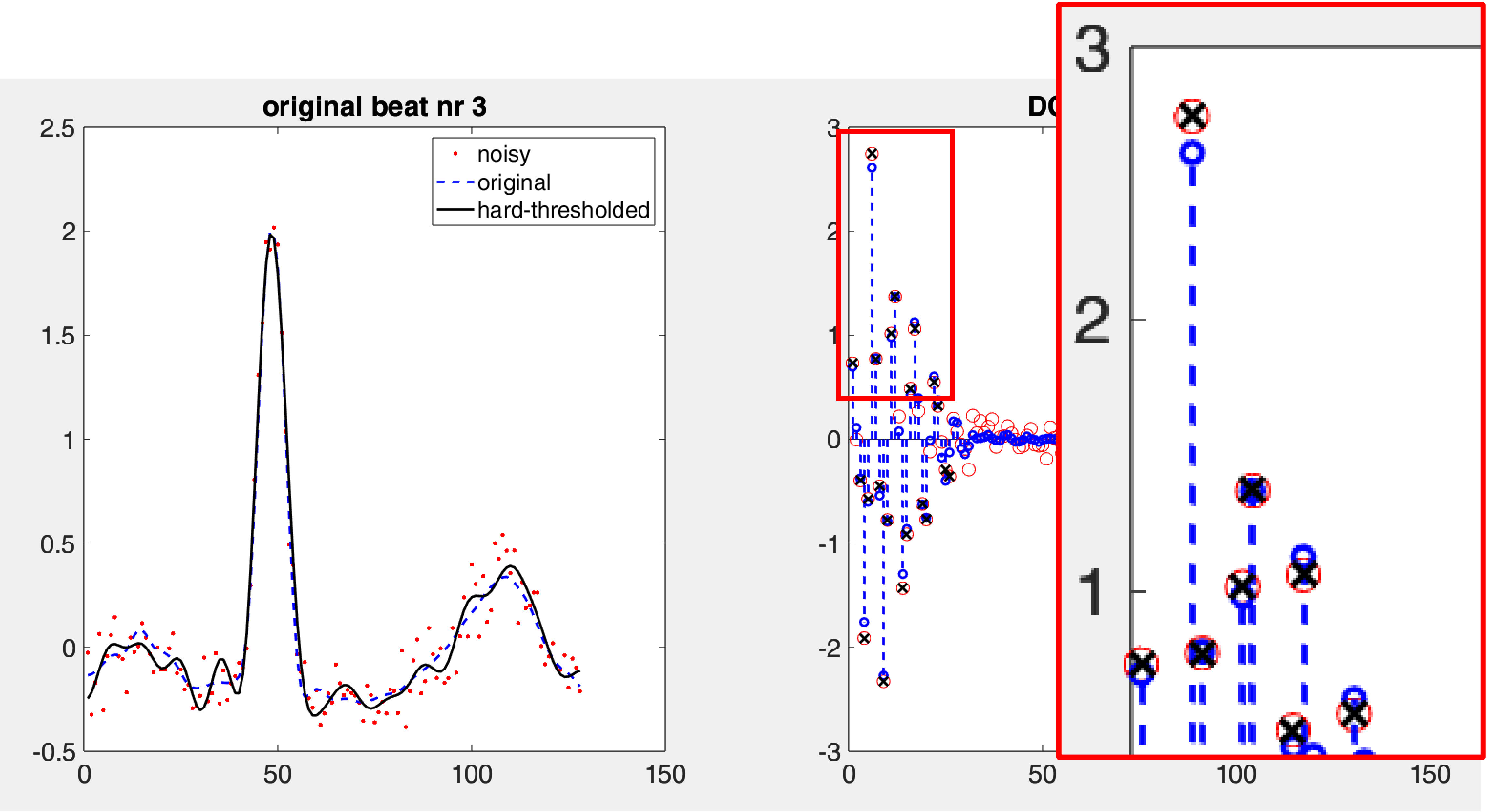

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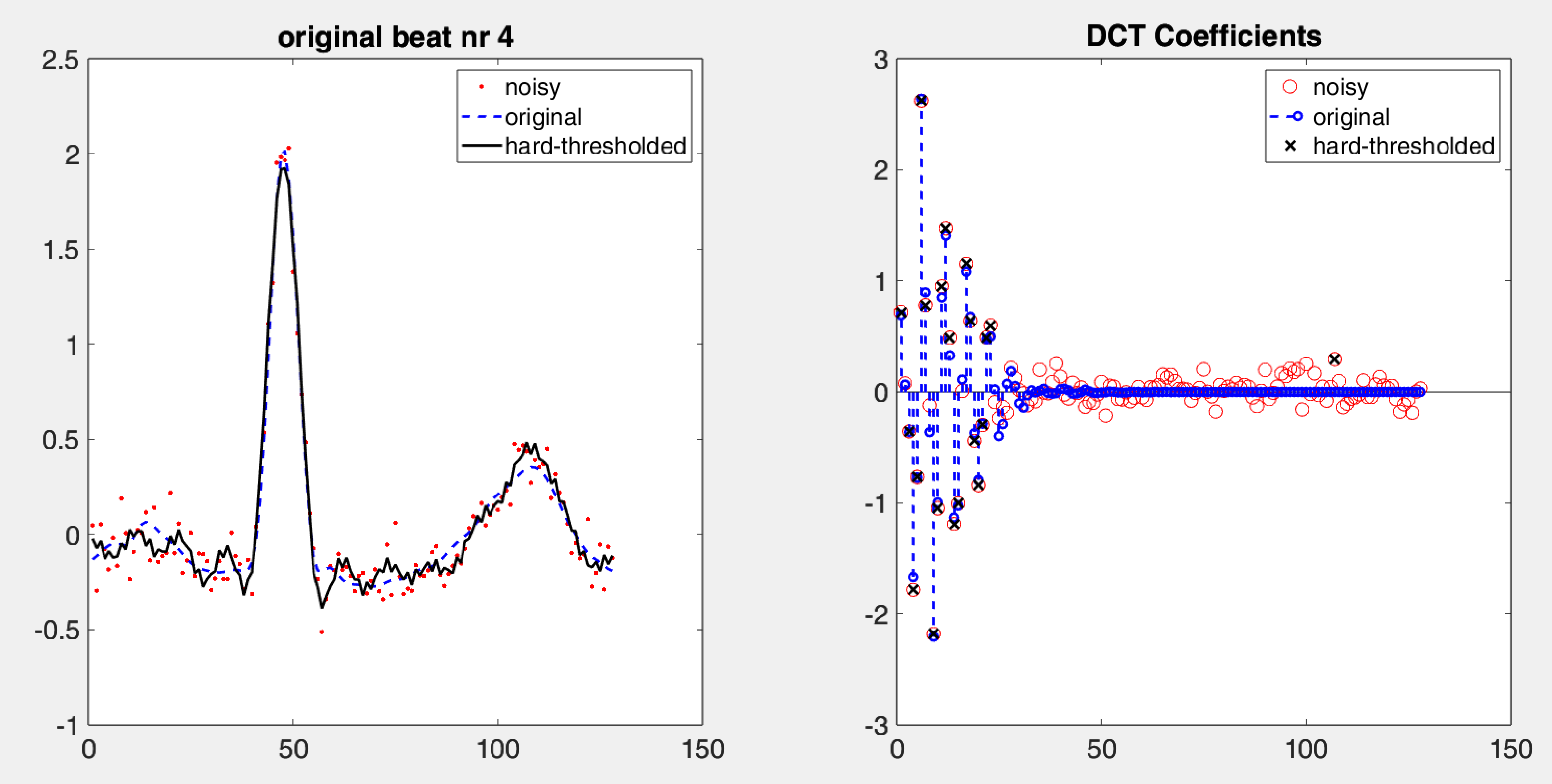
original beat nr 4
noisy
original
hard-thresholded
2.5
2
1.5
1
0.5
0
-0.5
-1
0
50
100
150
DCT Coefficients
noisy
original
hard-thresholded
3
2
1
0
-1
-2
-3
0
50
100
150

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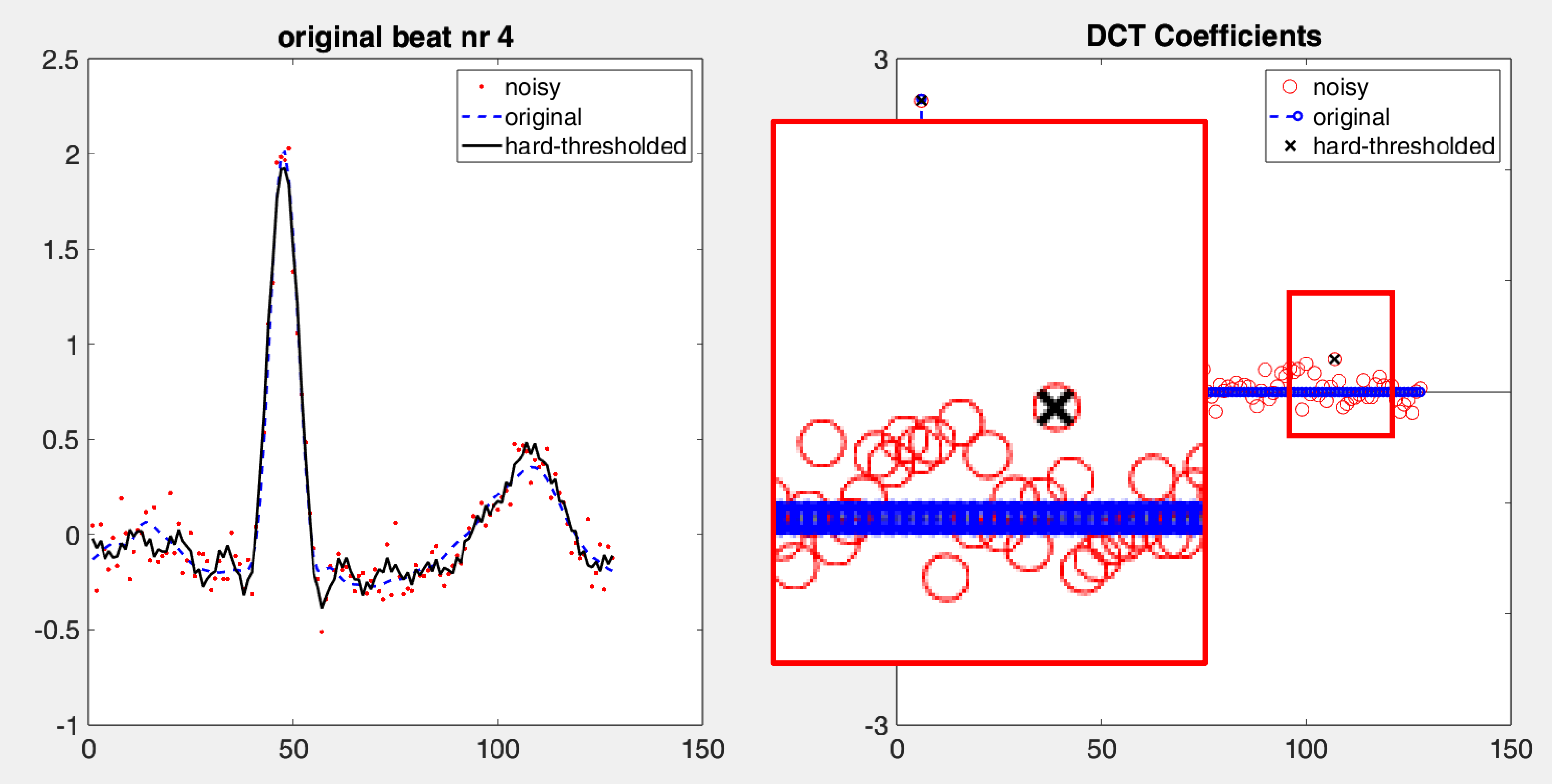
original beat nr 4
noisy
original
hard-thresholded
2.5
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1.5
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-1
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