Learning Sparse Representations for Image and Signal Modeling

PhD Course, DEIB 2023

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Assignments

Different ways of solving linear systems

There are different options for solving linear systems

- Construct an $N \times N$ matrix A corresponding to the 1-d Discrete Cosine Transform (DCT) for an N-dimensional vector. Choose a random x_0 and compute $s = Ax_0$.
- Recover x_0 from s (to investigate different solutions, ignore the fact that A' = A^-1):
  - solving the normal equations $A'A\,x = A'\,s$ and computing the inverse of $A'A$,
  - using the LU decomposition of A, (See Brendt's notes Section 1.2)
  - solving the linear system in Matlab using \
  - use conjugate gradient descent (see cgs in Matlab)

How do the computational cost and accuracy of these methods scale with N? Repeat the experiment with multiple different choices of s.

Other exercises

- Prove that $\frac{\partial}{\partial \mathbf{x}} ||\mathbf{x}||_2 = \frac{\mathbf{x}^T}{||\mathbf{x}||_2}$ and that $\frac{\partial}{\partial \mathbf{x}} ||\mathbf{x}||_2^2 = 2\mathbf{x}$
- Prove that the sum of convex functions (having the same domain) is also convex
- Prove that $||\mathbf{x}||_2^2$ and $||\mathbf{A}\mathbf{x} - \mathbf{s}||_2^2$ are convex functions
- Prove, by zeroing the derivatives, the key step in matching pursuit algorithms, i.e.,

$$\underset{z\in\mathbb{R}}{\operatorname{argmin}} \left|\left|\boldsymbol{b}_j\, z - \boldsymbol{s}\right|\right|_2^2 = \frac{\boldsymbol{b}_j'\boldsymbol{s}}{\left|\left|\boldsymbol{a}_j\right|\right|_2^2}$$

Minimum ℓ_0 algorithms

- Implement in separate functions the following algorithms:
- Matching Pursuit (MP): using stopping criteria on the sparsity, the maximum number of iterations and the reconstruction error
- Orthogonal Matching Pursuit (OMP): using as stopping criteria either the sparsity and/or the reconstruction error
  - Use OMP to solve the sparse coding for the DCT + delta case, using [D,C]
- Orthogonal Matching Pursuit (OMP-LS): using as stopping criteria either the sparsity and/or the reconstruction error
- Weak-MP: using the same stopping criteria of MP
- Thresholding (using Least Squares solutions to compute the projection on the selected subspaces), and the same stopping criteria of OMP
- See the next slides for hints about Weak-OMP and Thresholding

Minimum ℓ_0 algorithms

- Plug these functions in the image denoising framework developed in the previous assignment and test it using the dictionary learned from patches of natural images.

- Try OMP, MP, HT w.r.t. DCT basis, are these equivalent?

WEAK-MP (Reference)

Weak-MP implements a simpler and non-exhaustive sweeping step w.r.t. MP.

In each stage sweeping ends and as soon as one atom is found to reduce the residual by some threshold, i.e.,

$$\frac{(d_i' r^k)^2}{\|d_i\|_2^2} > \tau \|r^k\|_2^2$$

(or alternatively $\epsilon(j) < \tau \|r^k\|_2^2$)

Then, the selected atom is added to the support and everything proceeds as in MP

THRESHOLDING (Reference)

Thresholding:

- compute only once the errors that are computed at each step in MP

$$\epsilon(j) = \underset{z \in \mathbb{R}}{\text{argmin}} \left|\left|\boldsymbol{d_j}\, z - \boldsymbol{s}\right|\right|_2^2$$

- Sort the errors in a descending order
- Iteratively scan the atoms according to their sorted errors
  - Update the active set $\omega = \omega \cup \{j\}$
  - Compute as in OMP the LS projection on D_ω
- Stop when the residual is small enough: this allows to recover solutions that are sparser than the number of originally selected support