

Single-View Geometry

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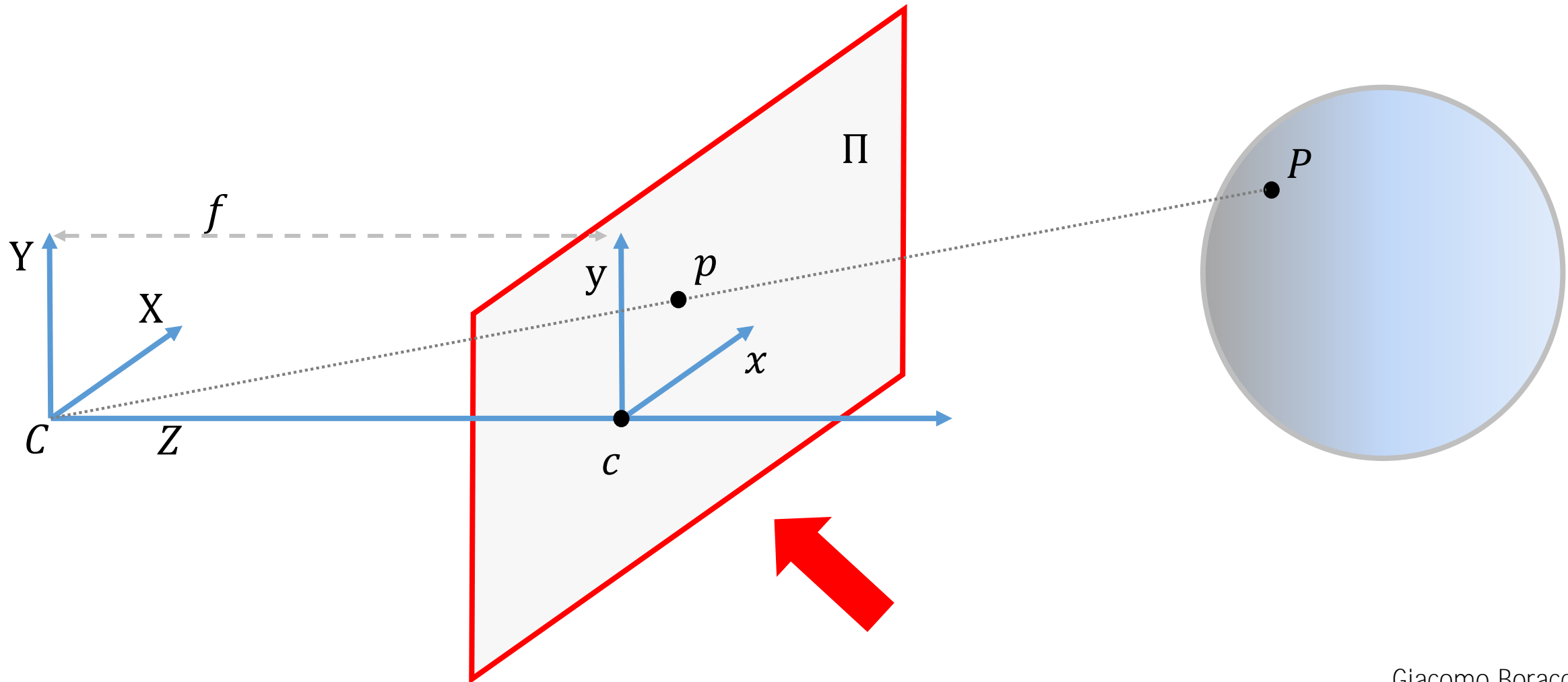
February 25th, 2020

USI, Lugano

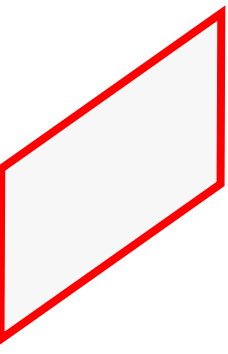
Book: HZ, chapter 2

Planar Geometry

Today we will study the geometry on the image plane Π , and in particular the properties of points and lines and their relationships



Points in \mathbb{P}^2



In homogeneous coordinates a point on Π corresponds to a triplet

$$\mathbf{x} = (x, y, w)^\top$$

We will always consider column vectors as these are more convenient in linear algebra operations

In homogeneous coordinates a point corresponds to a class of equivalence that includes all the scaled versions of \mathbf{x}

$$\mathbf{x} = (x, y, w)^\top = \left(\frac{x}{w}, \frac{y}{w}, 1\right)^\top$$

The latter representation is mapped in \mathbb{R}^2 by removing the last coordinate

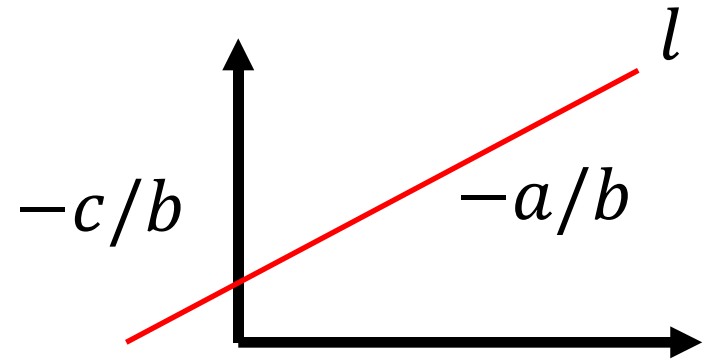
$$\left(\frac{x}{w}, \frac{y}{w}, 1\right)^\top \rightarrow \left(\frac{x}{w}, \frac{y}{w}\right)^\top \in \mathbb{R}^2$$

Lines in \mathbb{R}^2

In \mathbb{R}^2 the equation of a line l is

$$ax + by + c = 0$$

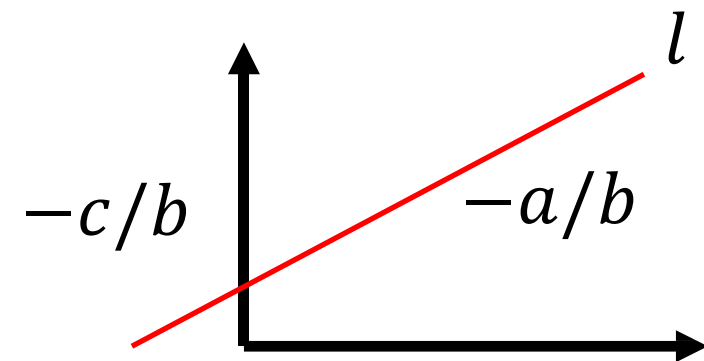
being $-a/b$ the angular coefficient and $-c/b$ the intercept



Lines in \mathbb{P}^2

Thus we can associate a point to each line

$$l \rightarrow [a, b, c]^T$$



This association is not one-to-one since $\lambda ax + \lambda by + \lambda c = 0$, $\lambda \neq 0$ identifies the same line but with different parameters

$$l \rightarrow [\lambda a, \lambda b, \lambda c]^T$$

Thus, representations $[\lambda a, \lambda b, \lambda c]^T$ and $[a, b, c]^T$ do coincide

Rmk: lines are naturally represented in \mathbb{P}^2

Rmk: this is the reason why a line has 3 coefficients but indeed only two degrees of freedom (gradient and intercept)

Rmk: the vector $[\mathbf{0}, \mathbf{0}, \mathbf{0}]^T$ does not correspond to any line

Incidence relation

A point $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} = (x, y)$ belongs to a line $l = [a, b, c]^T$ if and only if

$$ax + by + c = 0$$

The above relation can be written as

$$[a, b, c] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Where \cdot denotes the scalar product. Then

Property (incidence):

A point $\mathbf{x} \in \mathbb{P}^2$ lies on the line l if and only if

$$l^T \mathbf{x} = \mathbf{x}^T l = 0$$

Incidence relation

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Property (incidence):

A point $\mathbf{x} \in \mathbb{P}^2$ lies on the line l if and only if

$$l^T \mathbf{x} = \mathbf{x}^T l = 0$$

This is the reason why this number is set to 1 when moving from Euclidean to homogeneous coordinates

Intersection of lines

Property (intersection of lines):

The intersection of two lines l, m , their intersection $\mathbf{x} \in \mathbb{P}^2$ is

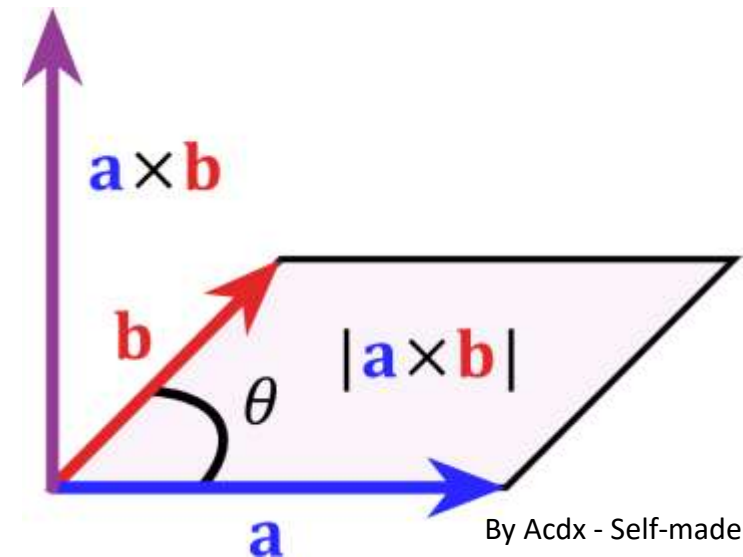
$$\mathbf{x} = l \times m$$

where \times denote the cross product of two 3-dimensional vectors.

Cross Product

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be two vectors, their cross product is a vector $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$

- That is perpendicular to the plane $\langle \mathbf{a}, \mathbf{b} \rangle$
- Has orientation of the right-hand rule
- Has length proportional to the area of the parallelogram spanned by the vectors, $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$



Cross Product

Rmk: the cross product can be also computed as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

being $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the versors of \mathbb{R}^3 and $|\cdot|$ the determinant

Rmk: the cross product is anti-commutative

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

But this is not an issue when we want to intersect two lines, since the result in the same point of \mathbb{P}^2 (equivalence up to a multiplication by -1)

Cross Product

Rmk: the cross product is distributive over addition

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Rmk: the cross product is compatible with scalar product

$$\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b})$$

Cross Product

Rmk: the cross product is not associative

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

Rmk: the triple scalar product identity follows by definition of cross product (which is orthogonal to both terms)

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

This also proves the fact the intersection of two lines $l \cap m$ is $\mathbf{x} = l \times m$

In fact \mathbf{x} belongs to both l and m since the incidence equation is satisfied
 $l \cdot (l \times m) = m \cdot (l \times m) = 0$

Line joining two points

Property: given $\mathbf{x}, \mathbf{y} \in \mathbb{P}^2$, the line l joining \mathbf{x} and \mathbf{y} is

$$l = \mathbf{x} \times \mathbf{y}$$

Rmk this can be verified by checking that both \mathbf{x} and \mathbf{y} belong to $\mathbf{x} \times \mathbf{y}$ through the incidence equation

Rmk: we have seen that the cross product is anti-commutative

$$\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$$

This is not an issue for the resulting lines, since these are intrinsically equivalent up to a multiplication by a scalar

An interesting property

An interesting property, given two points in homogeneous coordinates

$\mathbf{a} = [a_1; a_2; a_3]$ and $\mathbf{b} = [b_1; b_2; b_3]$ any linear combination $\lambda\mathbf{a} + \mu\mathbf{b}$, $\lambda, \mu \in \mathbb{R}$ belongs to the line joining \mathbf{a}, \mathbf{b}

Proof

Let $\mathbf{l} = \mathbf{a} \times \mathbf{b}$, for which holds $\mathbf{l}^\top \mathbf{a} = \mathbf{l}^\top \mathbf{b} = \mathbf{0}$ (incidence equation)

Then, $(\lambda\mathbf{a} + \mu\mathbf{b})\mathbf{l}^\top = \lambda\mathbf{a}\mathbf{l}^\top + \mu\mathbf{b}\mathbf{l}^\top = \mathbf{0} + \mathbf{0}$

Angles in the Euclidean Plane

In Euclidean geometry the angle between two lines is computed from the dot product of their normals. For the lines $l = (l_1; l_2; l_3)$, and $m = (m_1; m_2; m_3)$, the angle θ is such that

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We will see later interesting properties on projective plane

Intersection of parallel Lines

Consider two parallel lines $l = (a, b, c)$ and $m = (a, b, d)$, then their intersection is:

$$l \times m = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ a & b & d \end{vmatrix} = \mathbf{i}(bd - bc) - \mathbf{j}(ad - ac) + \mathbf{k}(ab - ab)$$

$$l \times m = \begin{bmatrix} bd - bc \\ ac - ad \\ 0 \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

That corresponds to the homogeneous point $\mathbf{x} = [b, -a, 0]^\top$

Rmk if we try to move to Euclidean coordinates we get $\left(\frac{b}{0}, \frac{a}{0}\right)$ which goes to infinity.

These are the ideal points or points at the infinity

Point at the Infinity

Definition: a point of \mathbb{P}^2 with the third coordinate equal zero is a point at the infinity (or ideal point)

Rmk: $l = [a, b, c]^T$ passes through the ideal point $l_\infty = [b, -a, 0]^T$

Rmk: ideal points can be seen as sort of *directions*

Rmk: \mathbb{P}^2 augments \mathbb{R}^2 by including *directions*

Rmk: finite points in \mathbb{P}^2 are those having the third coordinate $\neq 0$ and these corresponds to \mathbb{R}^2 up to a normalization factor

Line at the infinity ℓ_∞

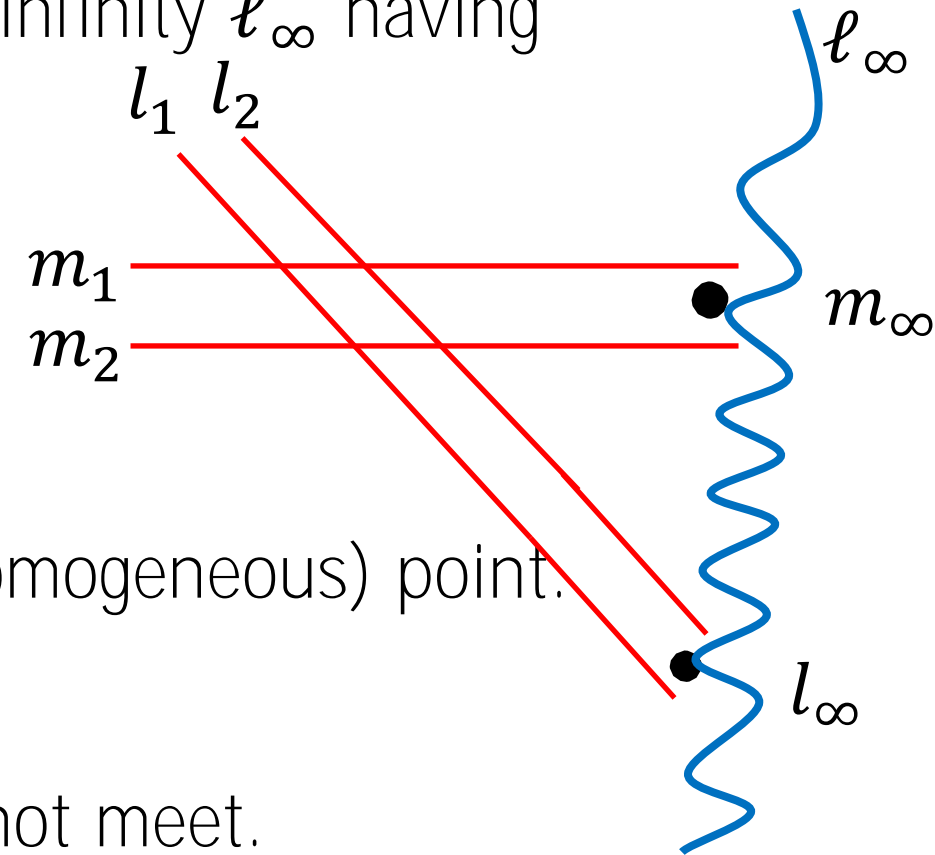
Property: all the ideal points lie in the line at the infinity ℓ_∞ having coordinates

$$\ell_\infty = [0; 0; 1]$$

Rmk: Thanks to ideal points

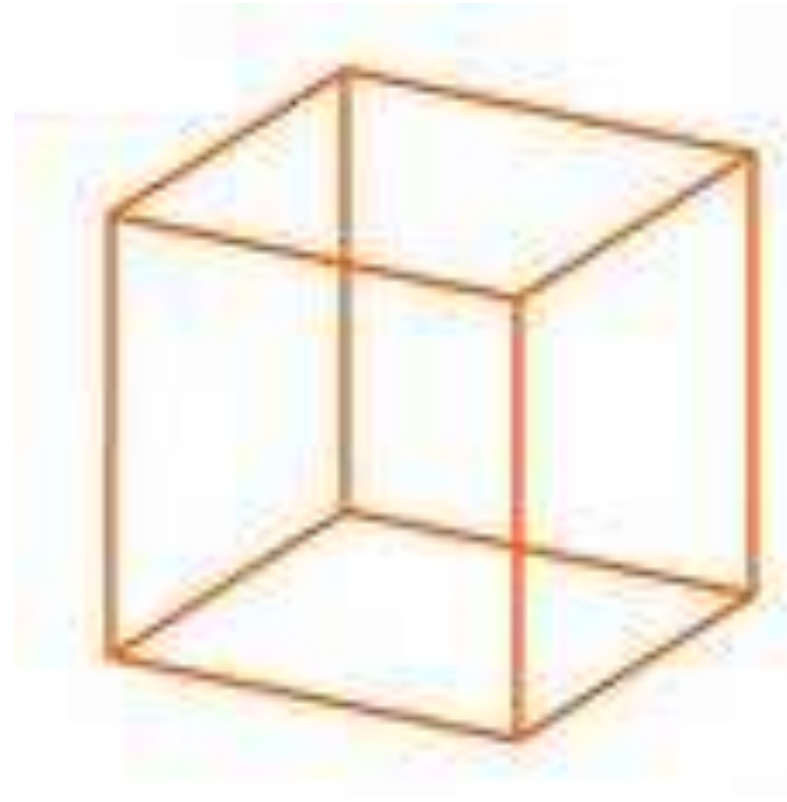
- any pair of lines in \mathbb{P}^2 intersect in a single (homogeneous) point.
- any pair of points of \mathbb{P}^2 lie a single line

This does not hold in \mathbb{R}^2 where parallel lines do not meet.



Example

Isometric cube, draw missing points as the intersection of lines parallel to this and passing through another point



A model for \mathbb{P}^2

Points of \mathbb{P}^2 are represented by all the rays of \mathbb{R}^3 through the origin, since all

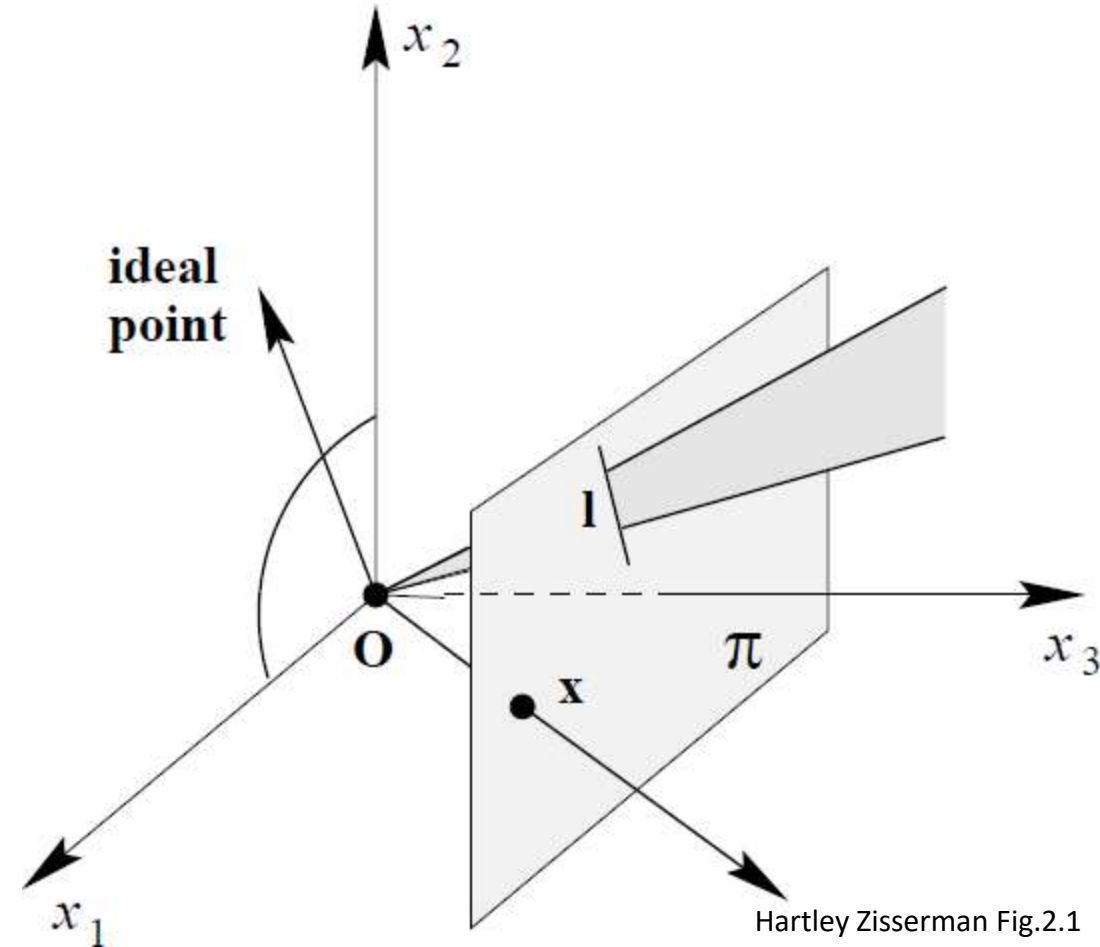
$$\mathbf{x} = \lambda[a; b; 1], \forall \lambda$$

corresponds to the same point

Lines of \mathbb{P}^2 are represented as planes passing through the origin

The Euclidean plane is the plane $\mathbf{x}_3 = 1$ and projection to Euclidean coordinates is computing the intersection between the ray and the plane Π , i. e., $\mathbf{x}_3 = 1$.

Lines lying in the plane $\mathbf{x}_3 = 0$ represent ideal points (directions) as these do not intersect Π .



Point – Line Duality in \mathbb{P}^2

Duality principle. *To any theorem of 2-dimensional projective geometry (\mathbb{P}^2) there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem.*

Theorem		Dual Theorem
- Point	→	- Line
- Line	→	- Point
- Belongs to	→	- Go through
- Go through	→	- Belongs to

E.g. The incidence equation, the line passing through two points which has the same formulation of the intersection between lines

Rmk: Note that is it not necessary to prove the dual of a given theorem once the original theorem has been proved

Linear Combinations in \mathbb{P}^2

An interesting property, given two points in homogeneous coordinates

$a = [a_1; a_2; a_3]$ and $b = [b_1; b_2; b_3]$ any linear combination $\lambda a + \mu b$, $\lambda, \mu \in \mathbb{R}$ belongs to the line joining a, b

Proof

Let $l = a \times b$, for which holds $l'a = l'b = 0$ (incidence equation)

Then, $(\lambda a + \mu b)l' = \lambda a l' + \mu b l' = 0 + 0$

Single-View Geometry

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February 28th, 2020

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Book: HZ, chapter 4

Outline

- Trasformations in \mathbb{P}^2
- The Projective Space \mathbb{P}^3
 - Vanishing Points
- Affine Rectification

Transformations in \mathbb{P}^2

Homographies

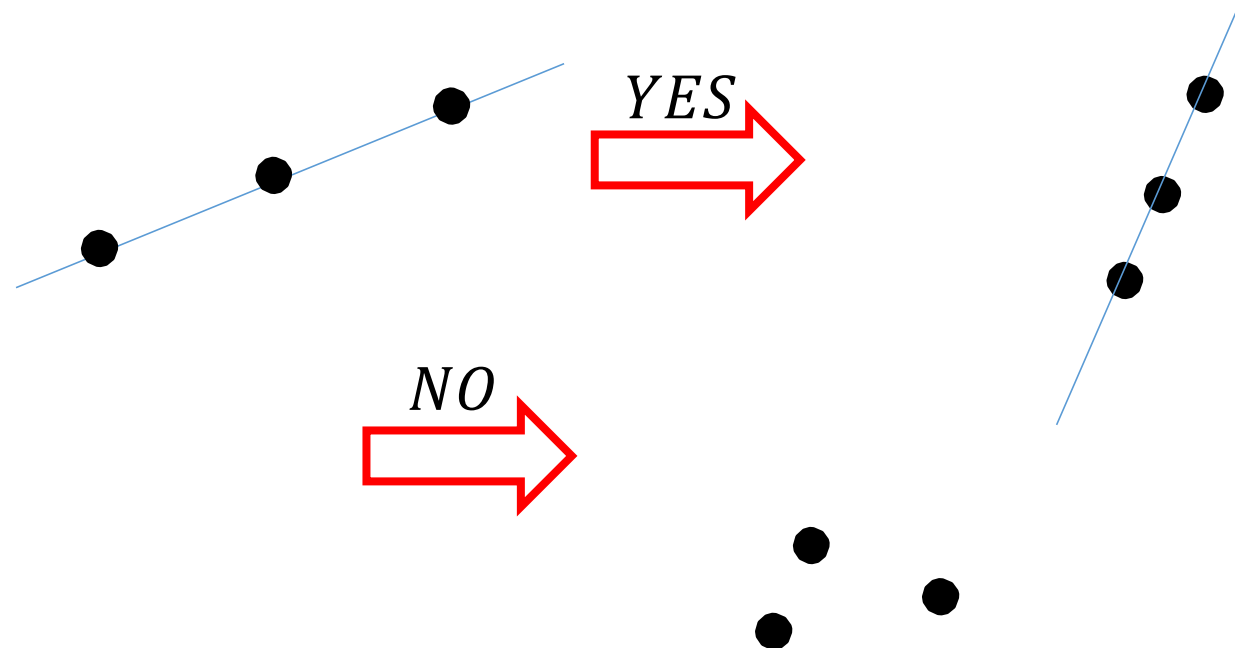
The most general transformation we consider in \mathbb{P}^2

Definition A *projectivity* is an invertible mapping h from \mathbb{P}^2 to itself such that three points \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 lie on the same line if and only if $h(\mathbf{x}_1)$, $h(\mathbf{x}_2)$ and $h(\mathbf{x}_3)$ do.

Rmk this property is called collinearity

Alternative names

- *Collineation*
- *Projective transformation*
- *Homography*



Homographies

Theorem 2.10. *A mapping $\mathbf{h}: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix \mathbf{H} such that for any point in \mathbb{P}^2 represented by a vector \mathbf{x} it is true that $\mathbf{h}(\mathbf{x}) = \mathbf{H}\mathbf{x}$*

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix}$$

Each and every linear mapping in \mathbb{P}^2 is an homography, and only linear mapping are homographies

Homographies and Points in \mathbb{P}^2

From theorem follows

$$\mathbf{x}' = H\mathbf{x}$$

Rmk: if we scale both \mathbf{x}' and \mathbf{x} by arbitrary factors the relation holds since we are in \mathbb{P}^2

$$\mathbf{x}' = \lambda H\mathbf{x}, \quad \forall \lambda \in \mathbb{R} \setminus \{0\}$$

Thus, H has 9 entries but only 8 degrees of freedom, since only the ratio between the elements counts. H is said to be an homogeneous matrix.

Homographies and Lines in \mathbb{P}^2

An homography transform each line \mathbf{l} in a line \mathbf{m} such that:

$$\mathbf{m} = (H^{-1})^T \mathbf{l}$$

We say that points transform *contravariantly* and lines and conics transform *covariantly*.

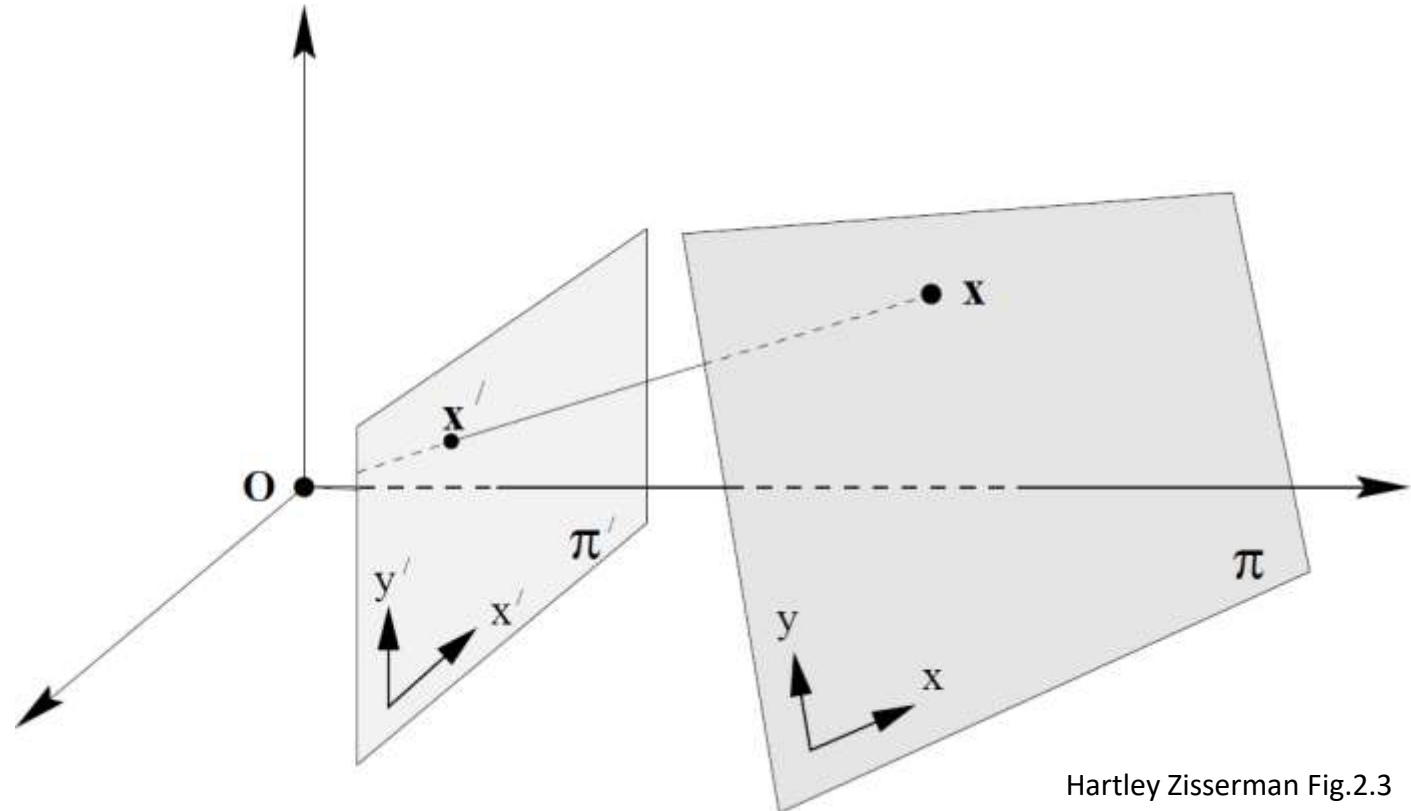
Mapping between planes: homography

Mapping between planes induced by a central projection is an homography, as this preserves collinearity

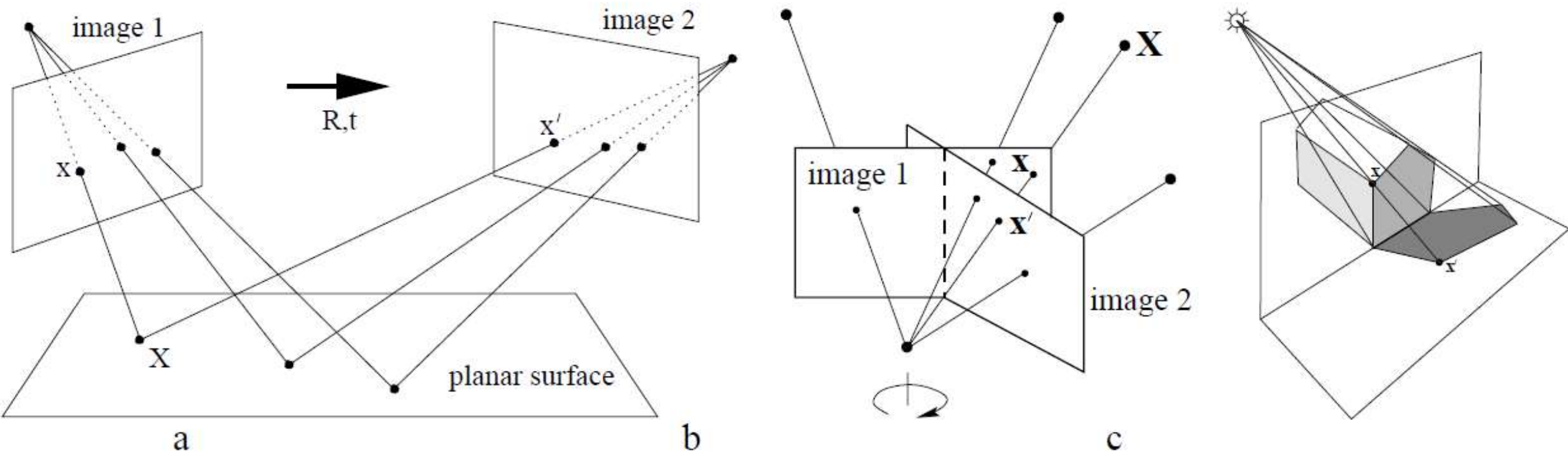
If a coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the *central projection* mapping may be expressed by

$$\mathbf{x}' = H\mathbf{x}$$

where H is a non-singular 3×3 matrix



Other examples where homographies apply



A Hierarchy of Transformations

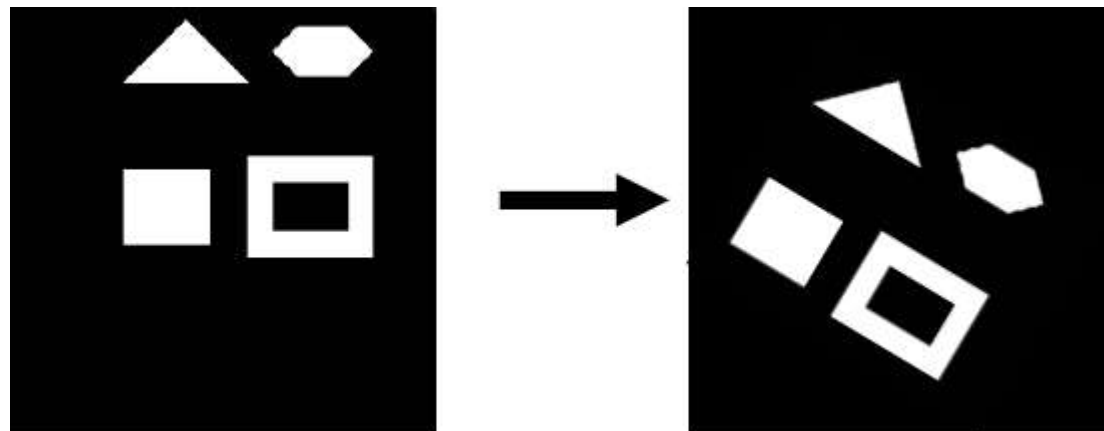
Linear Transformation

We consider linear transformations of \mathbb{P}^2 ,

- these can be expressed as a 3×3 matrix H ,
- the homogeneous constraint applies

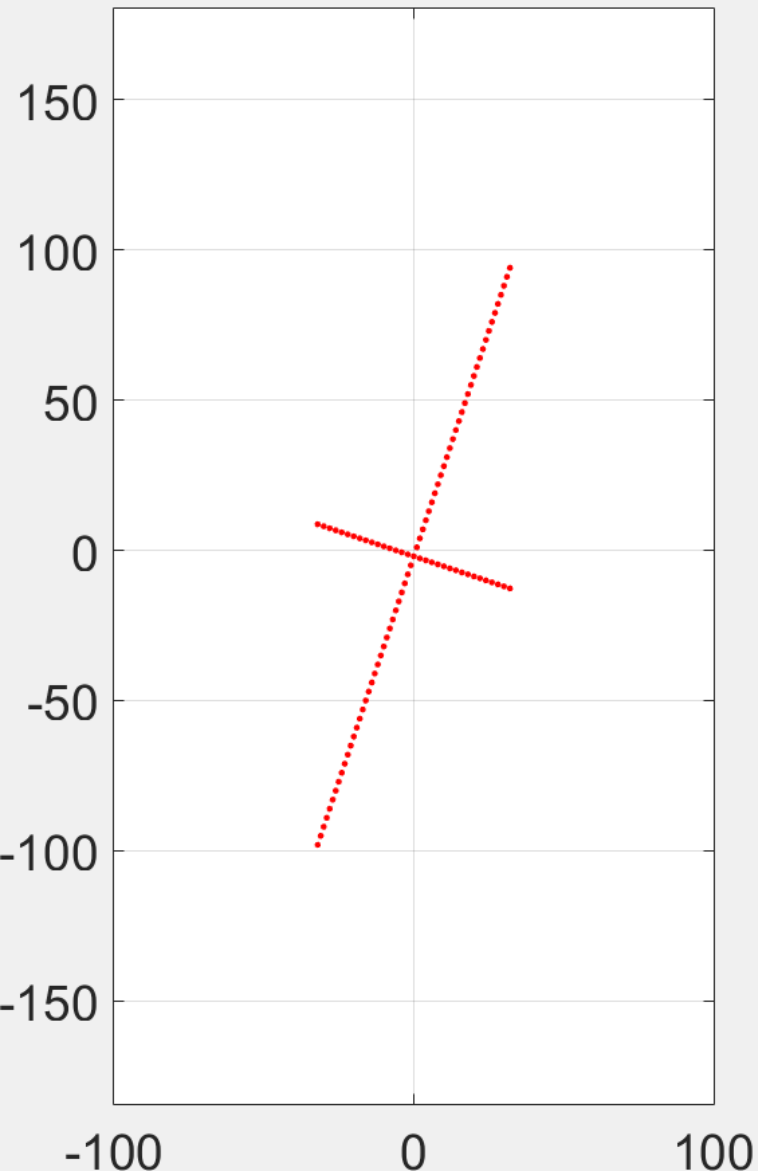
Depending on the structure of H there are different class of transformations with a different number of degrees of freedom

Isometries

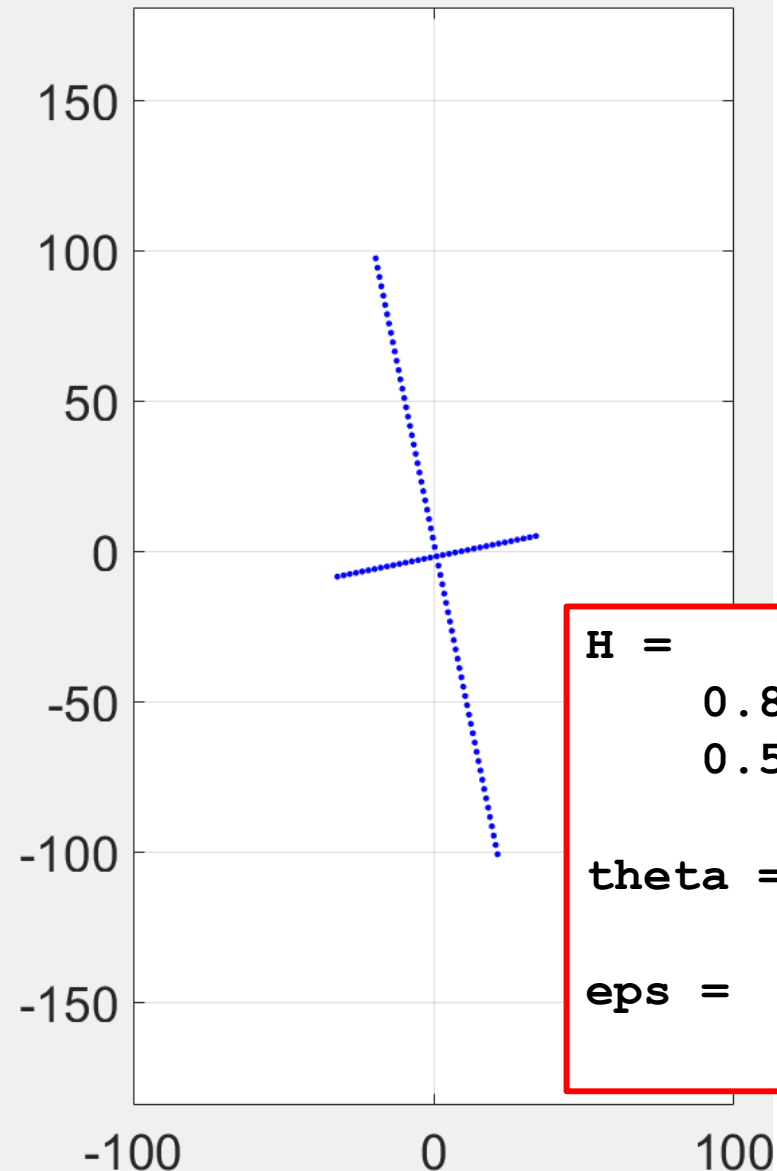


Isometries

input points



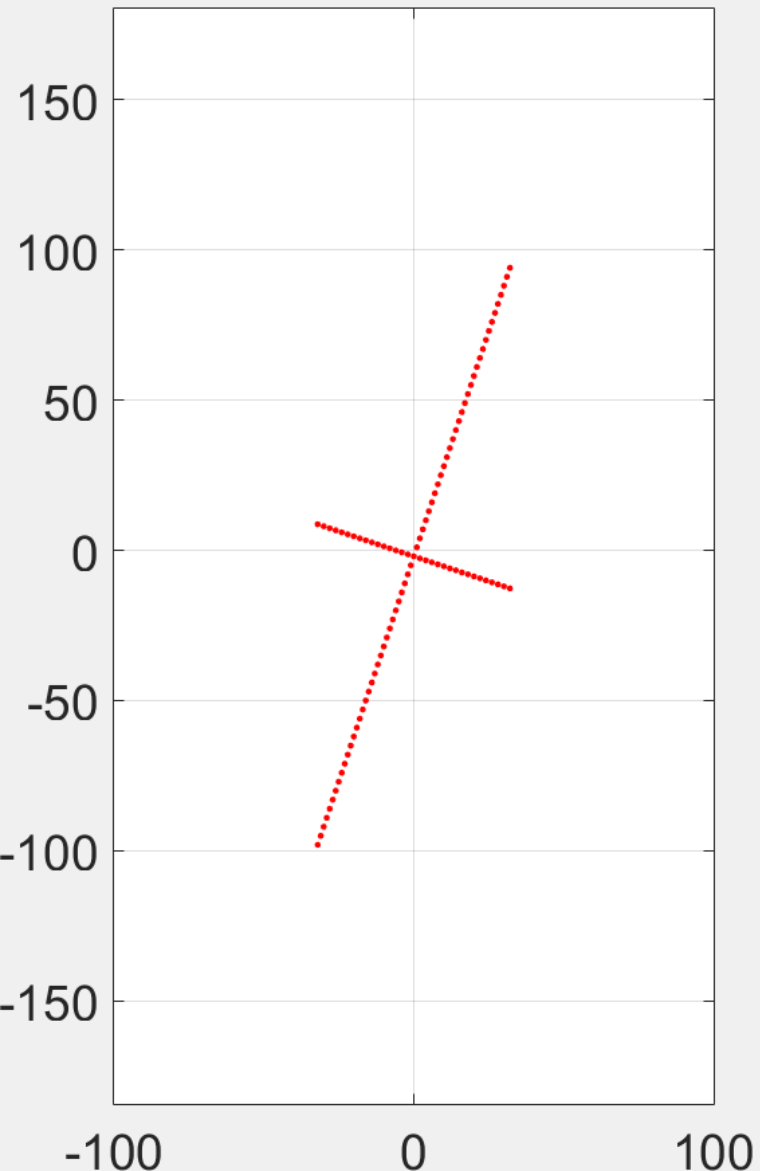
output points



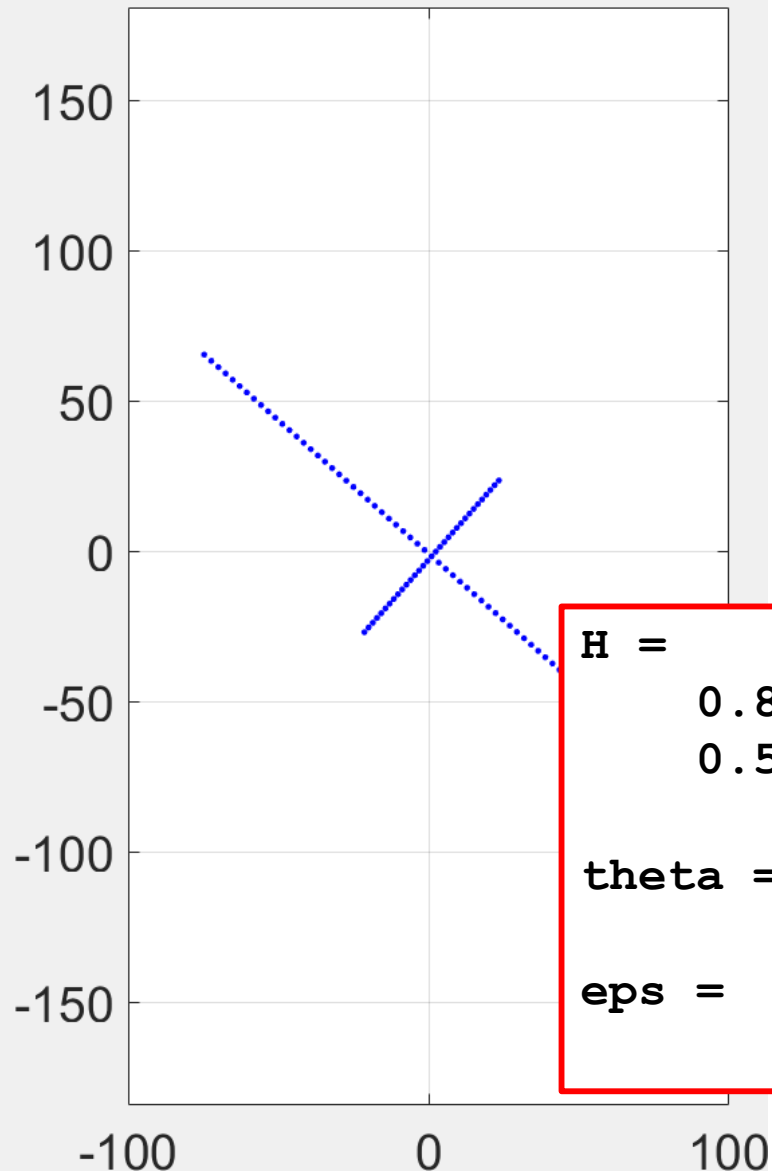
```
H =  
  0.8660  -0.5000  0  
  0.5000  0.8660  0  
  0  0  1.0000  
theta =  
  30  
eps =  
  1
```

Isometries

input points



output points



```
H =  
    0.8660    -0.5000     0  
    0.5000     0.8660     0  
         0         0     1.0000  
  
theta =  
    30  
  
eps =  
    -1
```

Isometries

Isometries can be written as

$$\mathbf{x}' = H_I \mathbf{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon \cos \theta & -\sin \theta & t_x \\ \epsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $\epsilon = \pm 1$.

Isometries

Isometries can be written as

$$\mathbf{x}' = H_I \mathbf{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $\mathbf{R} \in O(2)$ is a rotation matrix, i.e. an orthogonal matrix

$$\mathbf{R}^\top \mathbf{R} = \mathbf{R} \mathbf{R}^\top = I_2$$

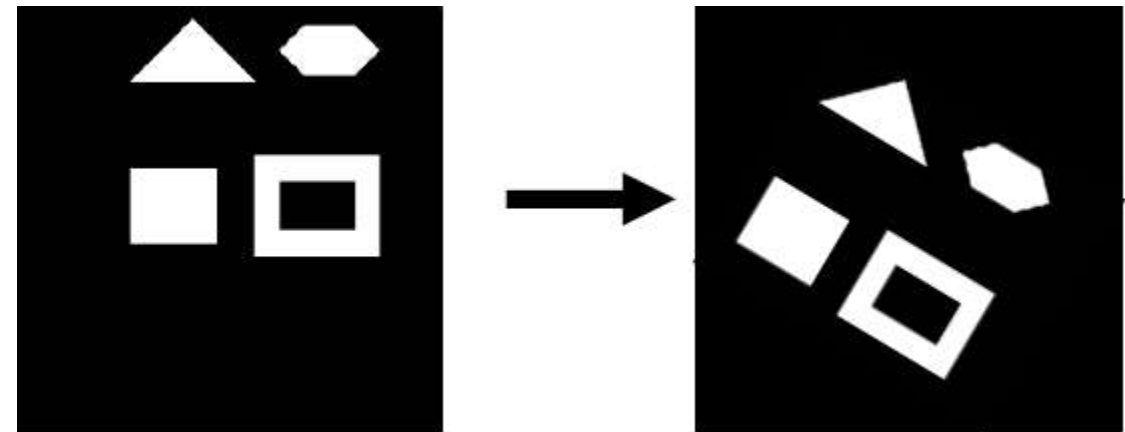
And $\mathbf{t} \in \mathbb{R}^2$ is a translation vector .

Pure rotation $\mathbf{t} = [\mathbf{0}; \mathbf{0}]$, pure translation $\mathbf{R} = I_2$

Isometries: Invariants

Isometries describe a 2D motion of a rigid object

Isometries preserves the angles, the distances, the areas.



Isometries: Remarks

Isometries have three degrees of freedom: the rotation angle θ and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon \cos \theta & -\sin \theta & t_x \\ \epsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Isometries can be estimated from two point correspondences

Rmk When $\epsilon = 1$ the isometry preserves the orientation, when $\epsilon = -1$ the isometry reverses the orientation.

For instance $\text{diag}(-1, 1, 1)$ is a reflection

Isometries: Remarks

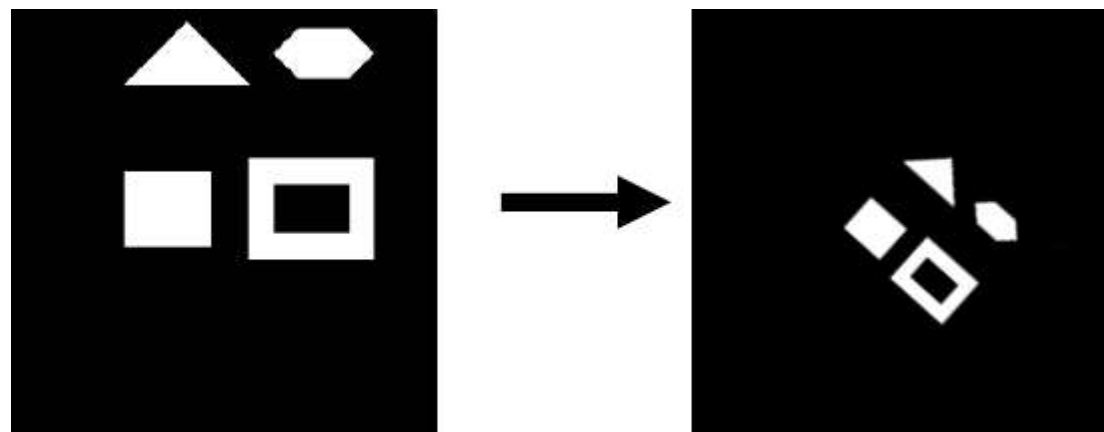
Isometries have three degrees of freedom: the rotation angle θ and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon \cos \theta & -\sin \theta & t_x \\ \epsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Isometries do not modify ideal points:

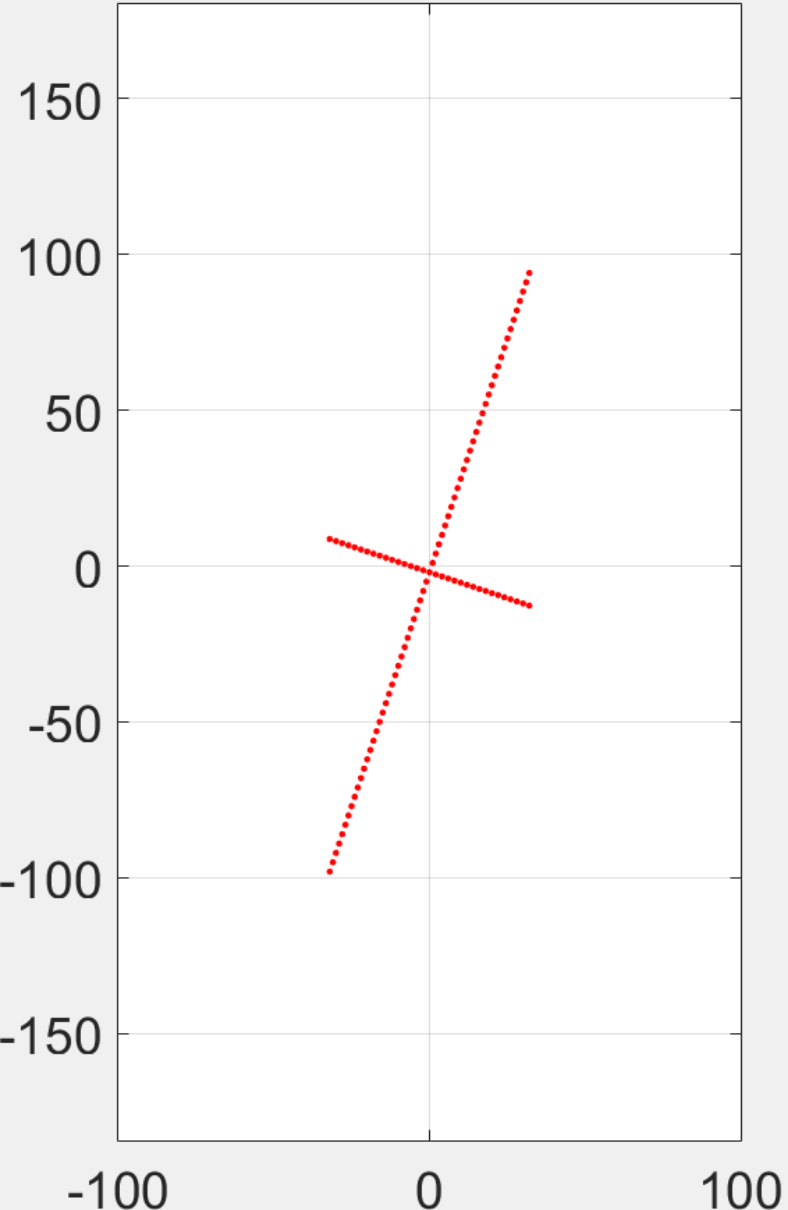
- Ideal points in \mathbb{P}^2 remain ideal points (they might change but always remain ideal points in ℓ_∞)
- Finite points in \mathbb{P}^2 remain finite points
- $\ell_\infty = H_I^{-T} \ell_\infty$

Similarities

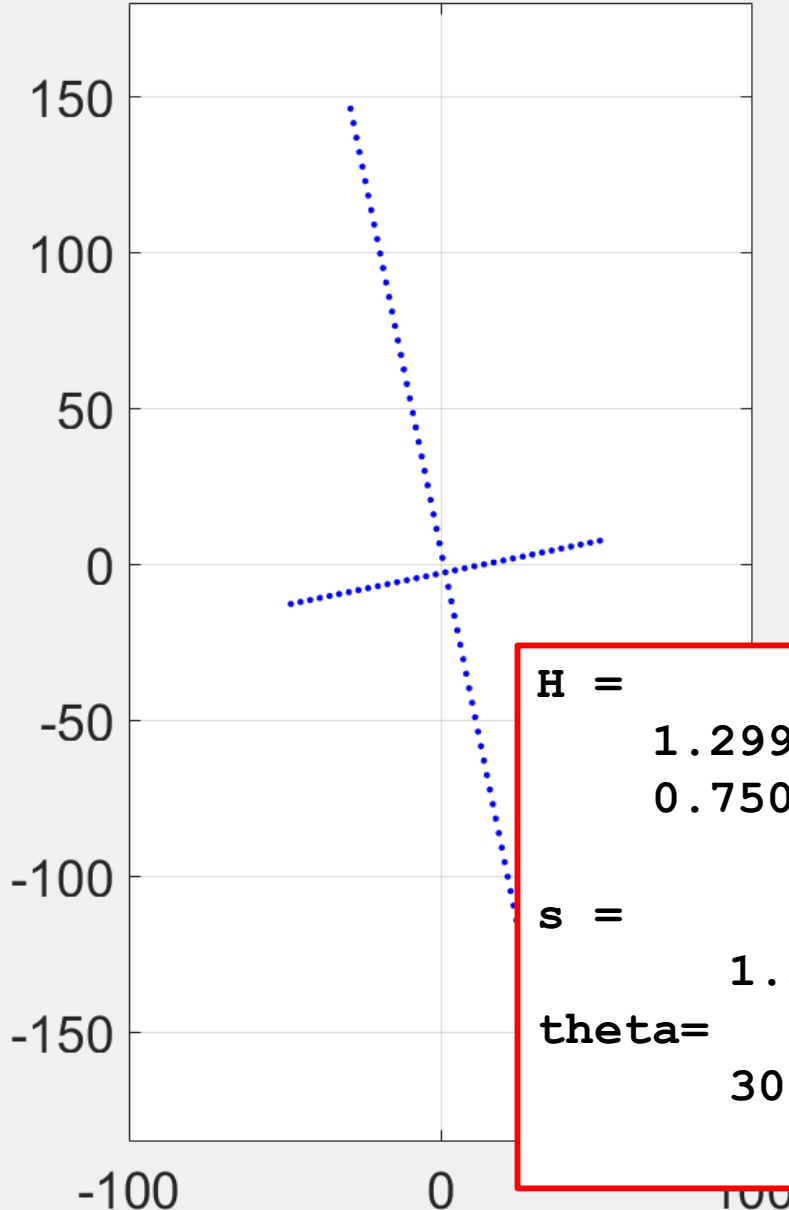


Similarities

input points



output points



```
H =  
  1.2990  -0.7500  0  
  0.7500  1.2990  0  
  0  0  1.0000  
s =  
  1.5  
theta=  
  30
```

Similarities

Similarities can be written as

$$\mathbf{x}' = H_S \mathbf{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $s \in \mathbb{R}$

Rmk Similarities can be seen as an isometry composed with an isotropic scaling of the axis

Similarities

Similarities can be written as

$$\mathbf{x}' = H_S \mathbf{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where

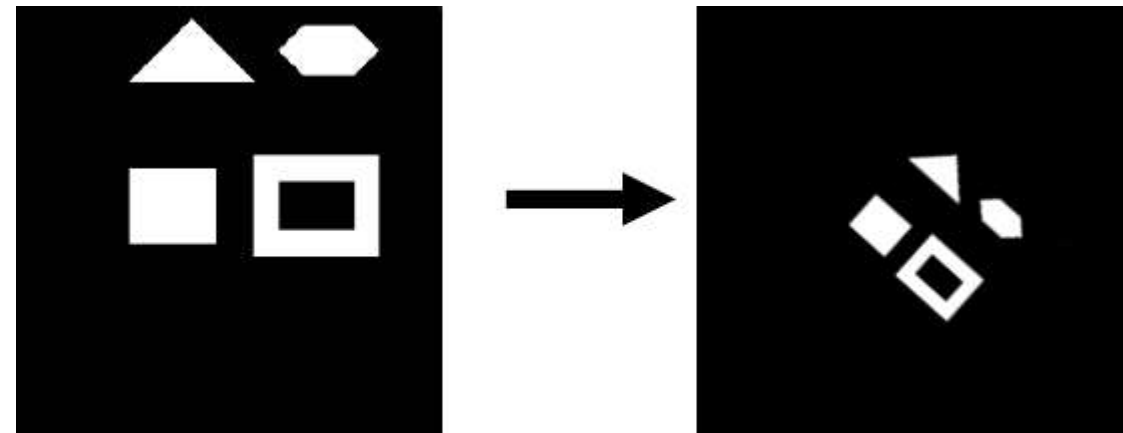
- $s \in \mathbb{R}$ is the scaling factor
- $\mathbf{R} \in O(2)$ is a rotation matrix,
- $\mathbf{t} \in \mathbb{R}^2$ is a translation vector.

Similarities: Invariants

Similarities are also known as *equi-form* transformations

Similarities preserves the angles and the shapes, while not the length and the areas.

The ratio between lengths and areas is preserved since the scaling factor cancels out



Similarities : Properties

Similarities have four degrees of freedom: the scaling factor s , the rotation angle θ and the translation vector $[t_x, t_y]$

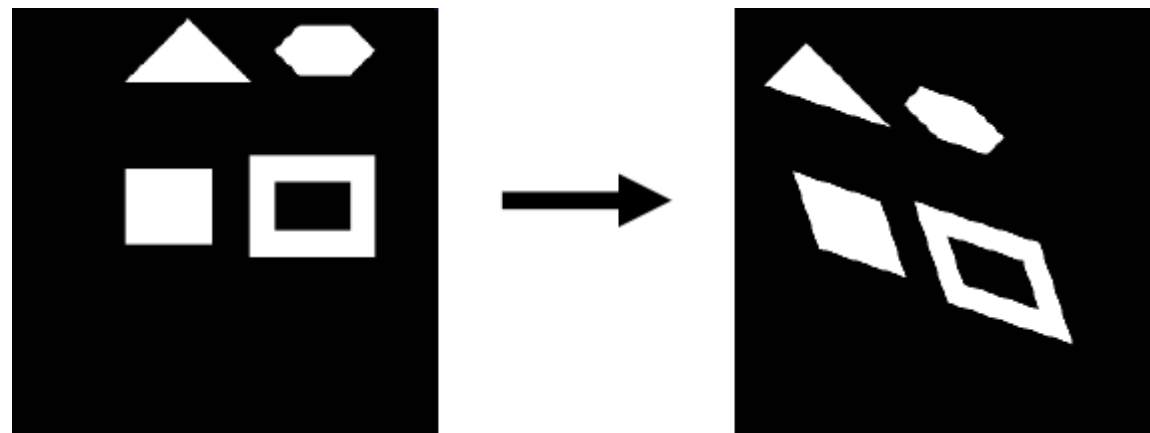
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s \cos \theta & -s \sin \theta & t_x \\ s \sin \theta & s \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk $\ell_\infty = H_S^{-1} \ell_\infty$, thus finite points remain finite points

Rmk Similarities can be estimated from two point correspondences

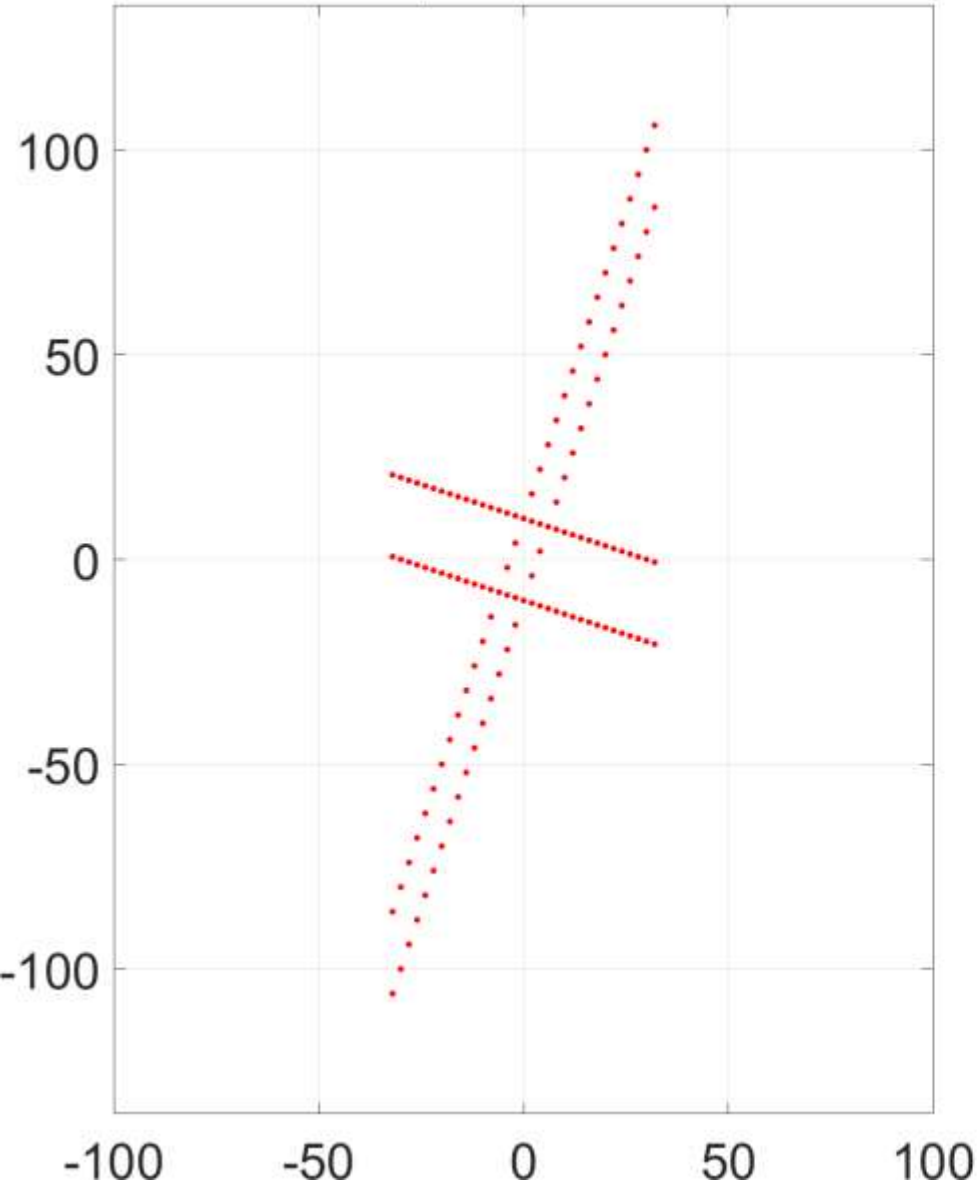
Metric structure means that the structure is preserved up to a scaling

Affinities

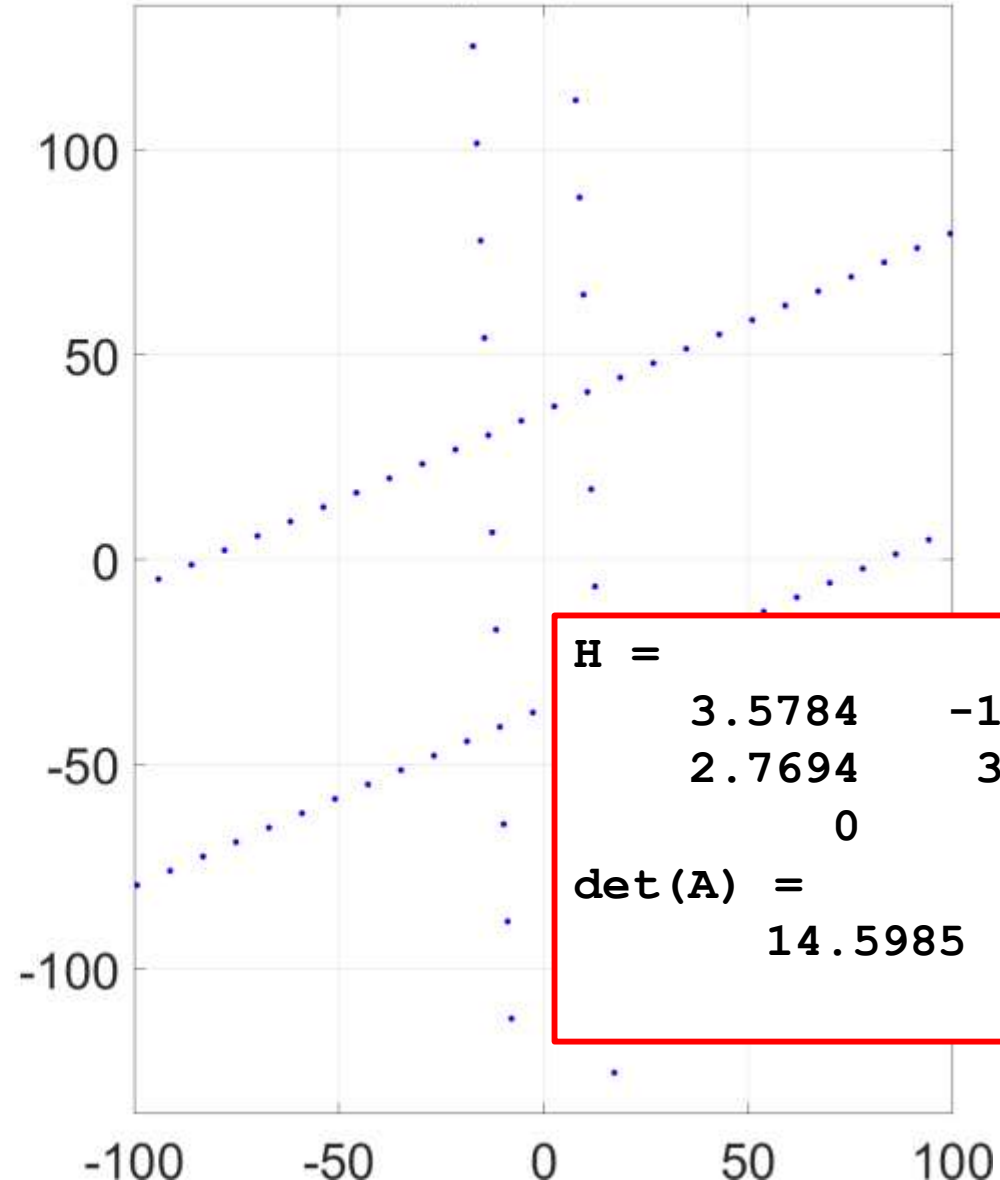


Affine Transformation

input points



output points



$H =$
 $\begin{bmatrix} 3.5784 & -1.3499 & 0 \\ 2.7694 & 3.0349 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}$
 $\det(A) =$
 14.5985

Affinities

Affinities can be written as

$$\mathbf{x}' = H_A \mathbf{x}$$

where

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & t_x \\ a_{2,1} & a_{2,2} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ is an invertible matrix

Affinities

Affinities can be written as

$$\mathbf{x}' = H_A \mathbf{x}$$

A more compact representation is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where

- $A \in \mathbb{R}^{2,2}$ is an invertible matrix
- $\mathbf{t} \in \mathbb{R}^2$ is a translation vector.

Rmk Affinities can be seen as a linear, non singular transformation of followed by a translation

Decomposition of Affinities

The linear part of the affine transformation can be seen as the composition of two fundamental transformations:

- rotation
- and non-isotropic scaling.

The affine matrix A can always be decomposed

$$A = R(\theta) R(-\varphi) D R(\varphi)$$

where $R(\theta)$ and $R(\varphi)$ are rotations by θ and φ respectively, and D is

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Decomposition of Affinities

This decomposition can be obtained via $SVD(A)$

$$A = UDV^T, \text{ being } UU^T = VV^T = I_2$$

$$A = U(VV^T)DV^T =$$

$$A = (UV)(V^T D V^T)$$

$$A = R(\theta) R(-\varphi) D R(\varphi)$$

The only “new” geometry, compared to a similarity, is the non-isotropic scaling.

Non-isotropic Scaling in Affinities

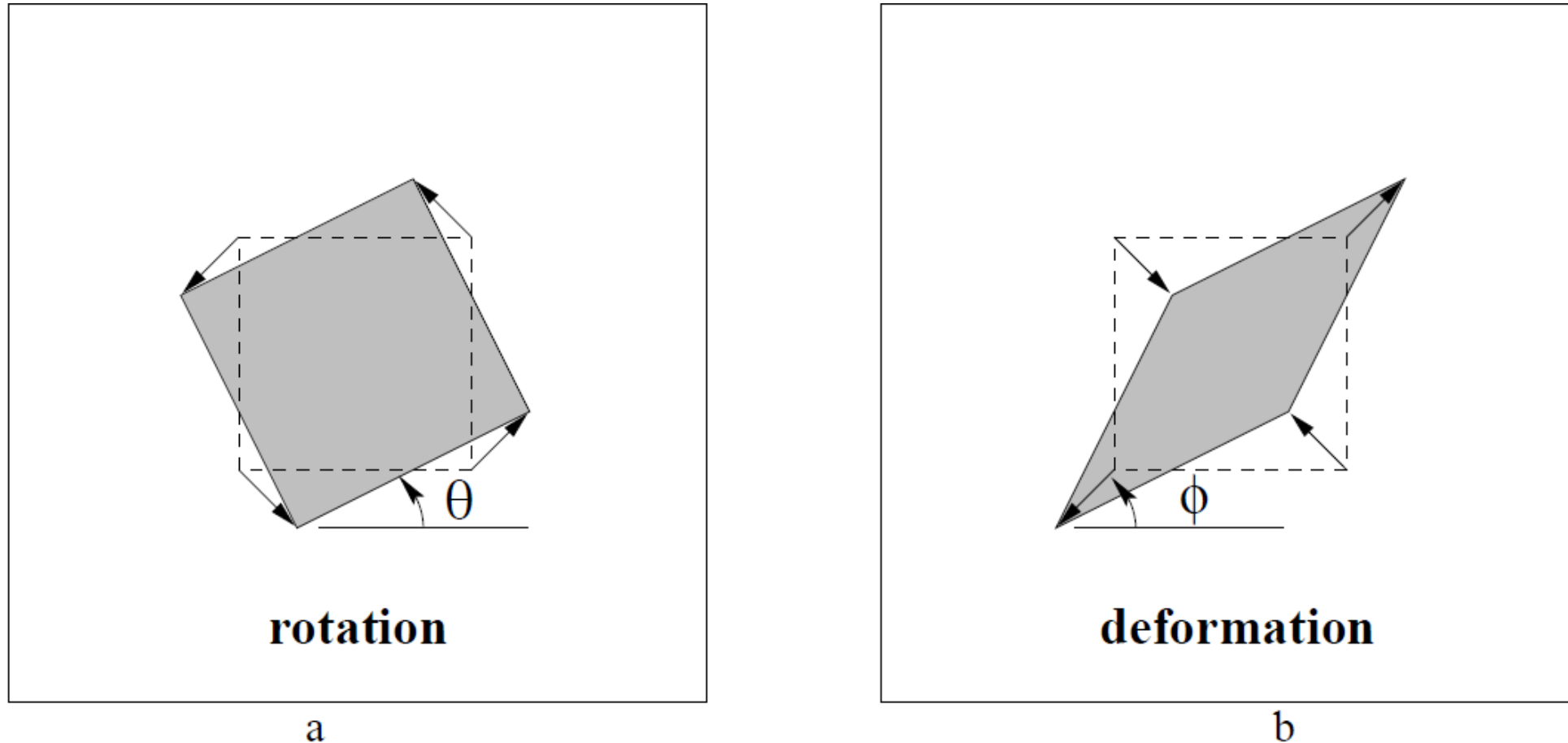


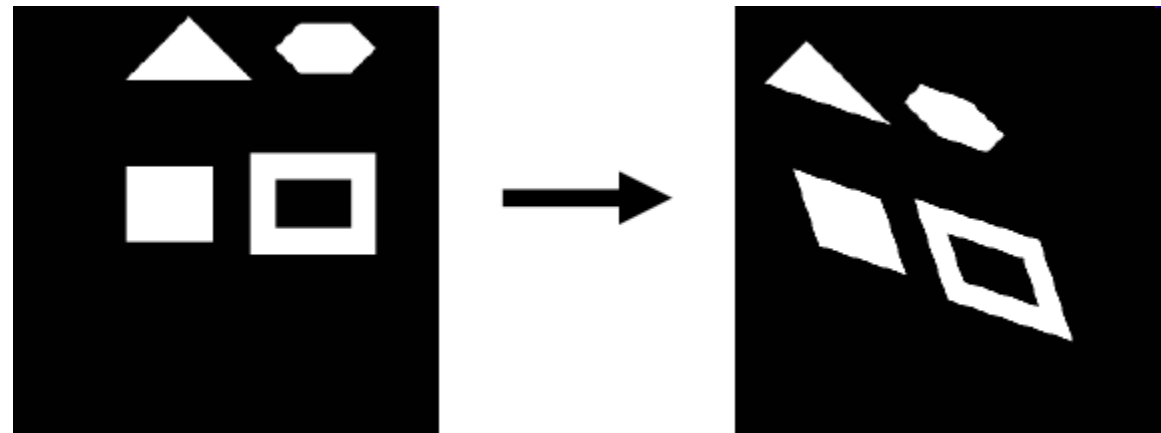
Fig. 2.7. **Distortions arising from a planar affine transformation.** (a) Rotation by $R(\theta)$. (b) A deformation $R(-\phi) D R(\phi)$. Note, the scaling directions in the deformation are orthogonal.

Affinities: Invariants

Due to the non-isotropic scaling, the similarity invariants of length ratios and angles between lines are not preserved under an affinity.

Affinities preserves

- Parallel lines, since $\ell_\infty = H_A^{-T} \ell_\infty$
- Ratio of lengths over parallel lines
- Ratio of areas (since each area is scaled of $\det(A)$, which cancels out)



Affinities: Properties

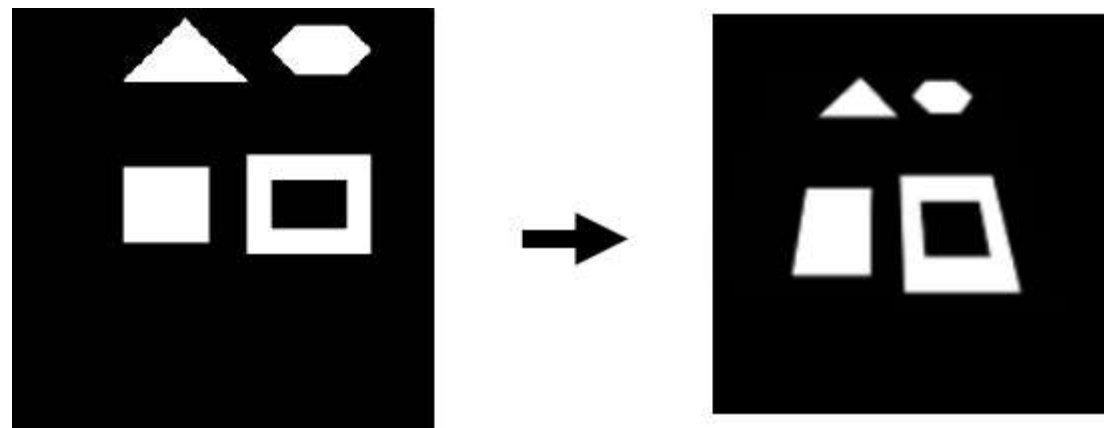
Similarities have six degrees of freedom: the four terms of \mathbf{A} and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & t_x \\ a_{2,1} & a_{2,2} & t_y \\ \boxed{0} & \boxed{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Similarities can be estimated from four point correspondences

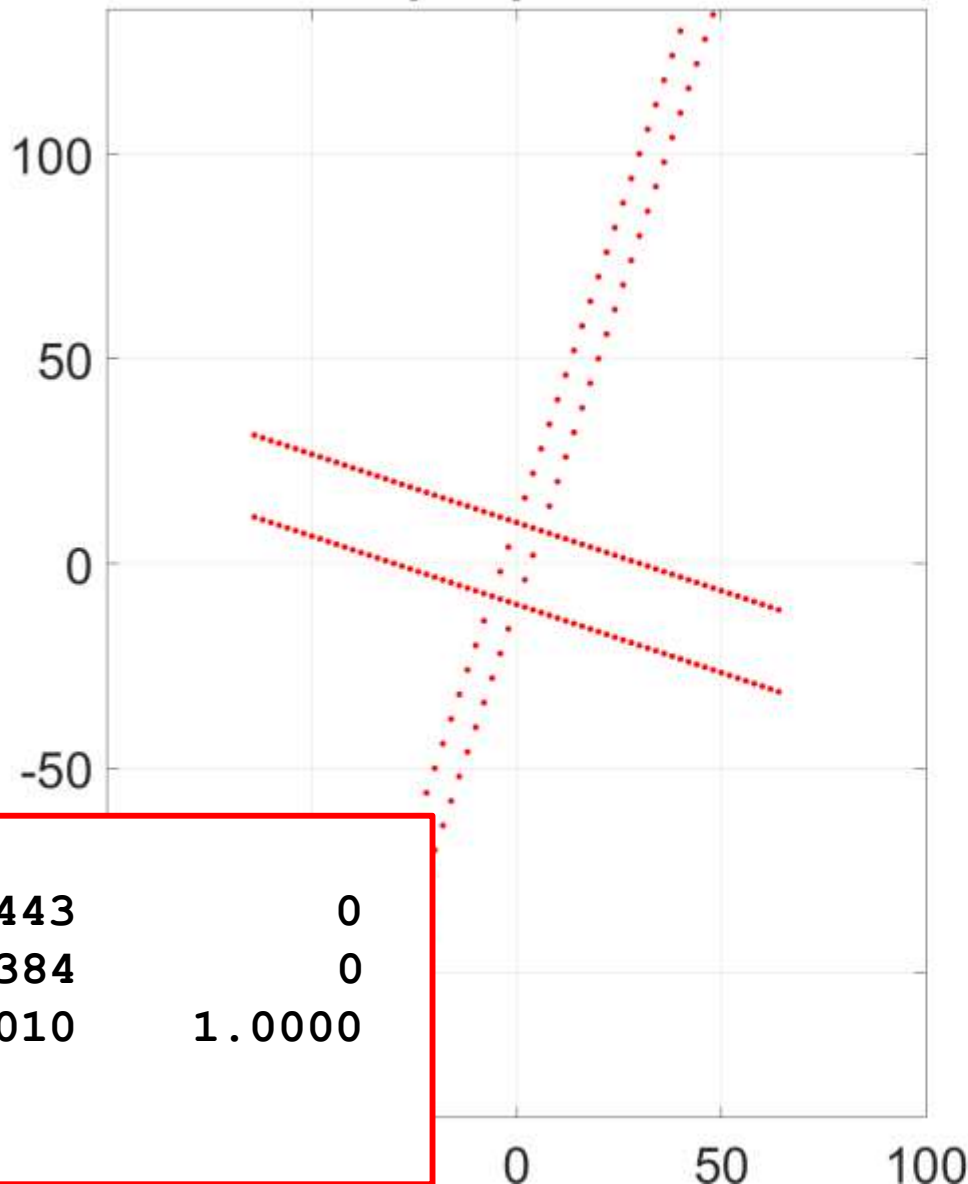
Rmk $\ell_\infty = H_A^{-T} \ell_\infty$, thus finite points remain finite points

Homographies

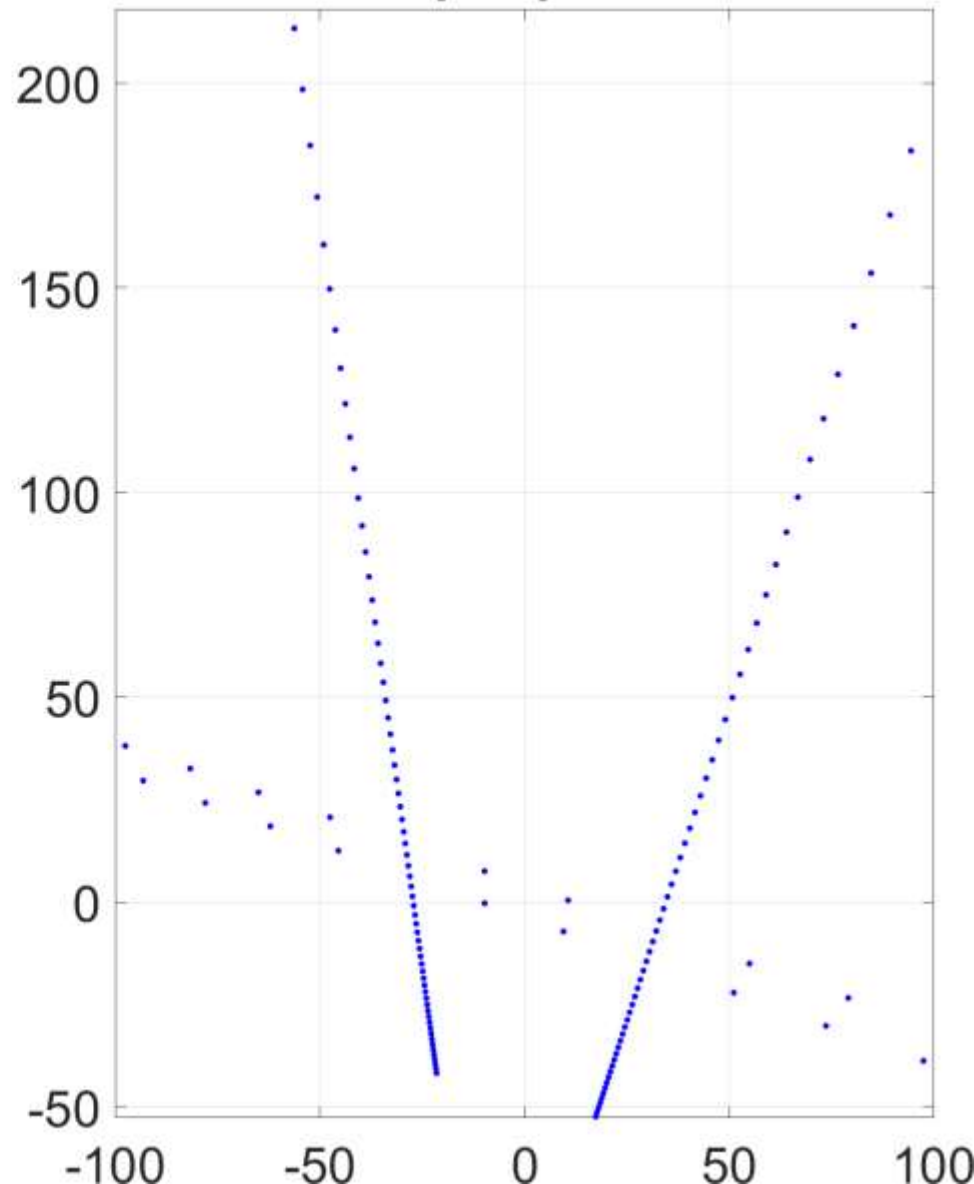


Projective Transformation

input points



output points



H =
-1.0689 -2.9443 0
-0.8095 1.4384 0
0.0100 0.0010 1.0000
det(A) =
-3.9209

Homographies

Homographies are the most general transformation *in homogeneous coordinates*, and can be written as

$$\mathbf{x}' = H_P \mathbf{x}$$

namely

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where $\mathbf{v} = [v_1; v_2]$ and $v \in \mathbb{R}$ (possibly zero)

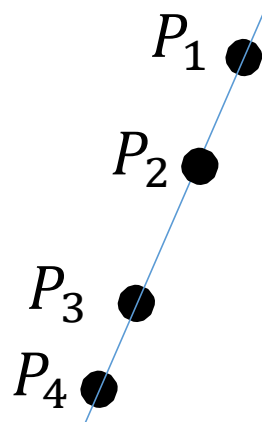
While this transformation applies in \mathbb{P}^2 , it is not always possible to scale H_P to have $v = 1$, as this might be zero.

Homographies: Invariants

Homographies generalize an affine transformation, which is the composition of a general non-singular linear transformation of *inhomogeneous* coordinates and a translation.

Invariants: the *cross ratio* of four collinear points: a ratio of lengths on a line is invariant under affinities, but not under projectivities. However, a ratio of ratios or *cross ratio* of lengths on a line is a projective invariant.

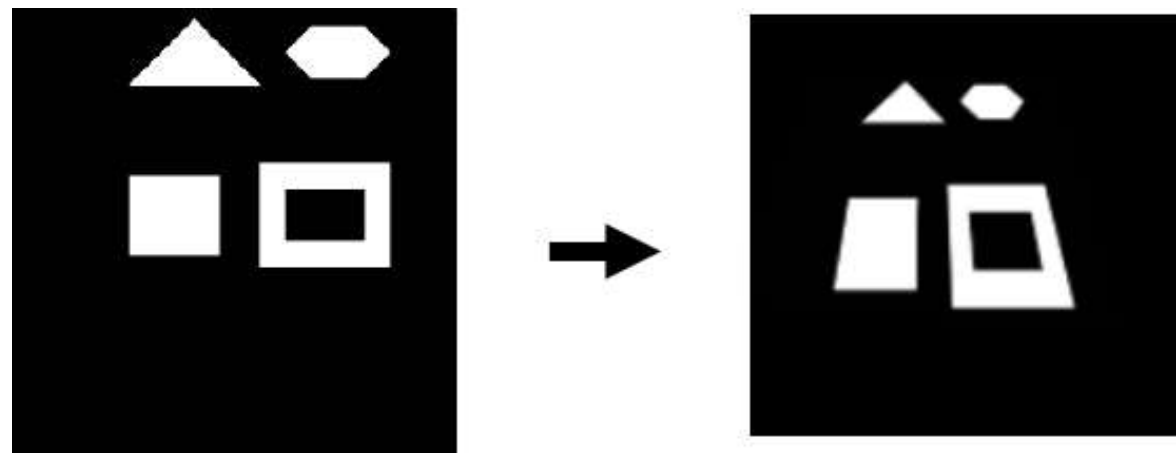
$$\frac{\|P_3 - P_1\| \|P_4 - P_2\|}{\|P_3 - P_2\| \|P_4 - P_1\|}$$



Homographies: Invariants

The matrix has nine elements with only their ratio significant, so the transformation is specified by eight parameters

A projective transformation between two planes can be computed from four point correspondences, with no three collinear on either plane.



Homographies: Properties

Similarities have eight degrees of freedom: the four terms of \mathbf{A} and the translation vector $[t_x, t_y]$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rmk Homographies can be estimated from four point correspondences

Rmk $\ell_\infty \neq H_P^{-\top} \ell_\infty$ when $\mathbf{v} \neq \mathbf{0}$, thus ideal points might become finite points. For instance given $\mathbf{x} = (x_1; x_2; 0)$, then

$$H_P \mathbf{x} = \begin{bmatrix} \dots \\ \dots \\ v_1 x + v_2 y + v_0 \end{bmatrix}$$

Homographies: Properties

Under a projective transformation ideal points may be mapped to finite points, and consequently ℓ_∞ is mapped to a finite line.

However, if the transformation is an affinity ($\mathbf{v} = [\mathbf{0}, \mathbf{0}]$), then ℓ_∞ is not mapped to a finite line, but remains at infinity.

The image of l_∞ is

$$(H^{-1})^\top l_\infty$$

$$\begin{bmatrix} A^{-\top} & \mathbf{v} \\ \mathbf{t}^{-\top} A^{-\top} & \mathbf{v} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v \end{bmatrix}$$

Affine Transformation and Line at Infinity

Rmk: l_∞ is not fixed pointwise under an affine transformation

Under an affinity a point on l_∞ (i.e., an ideal point) can be mapped to a different point on l_∞ . This is the reason why orthogonality is lost.

Affine Rectification



a


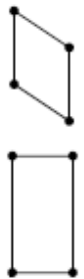

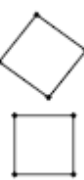


b


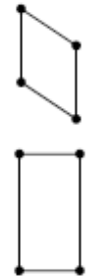
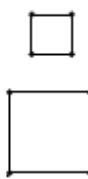
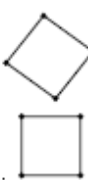


c

Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, l_∞ .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, I, J (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

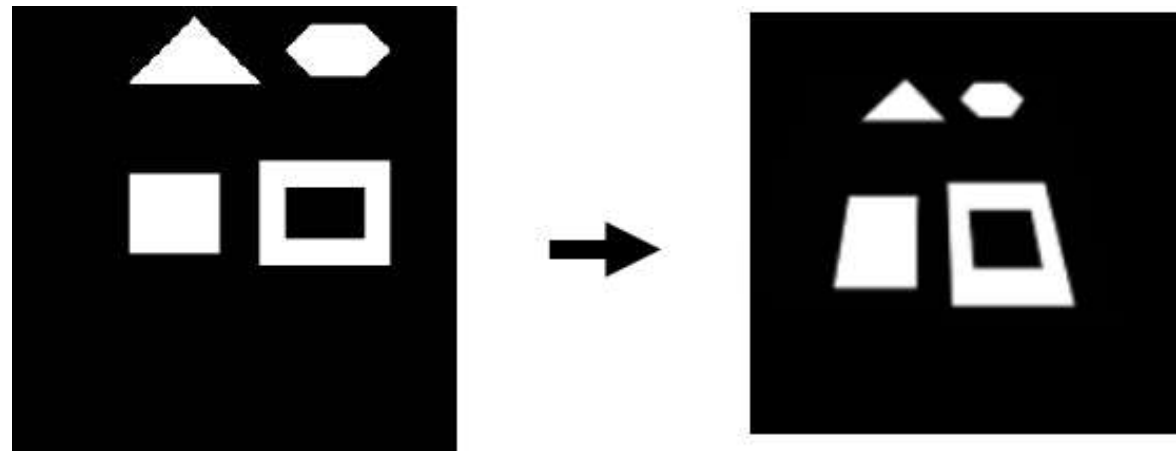
Summarizing

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<div style="background-color: yellow; padding: 5px; display: inline-block;"> <p><i>Transformations higher in the table can produce all the actions of the ones below</i></p> </div>
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, I, J (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Homographies: Properties

For a given affinity the areas are scaled of $\det(\mathbf{A})$ anywhere on the plane; and the orientation of a transformed line depends only on its initial orientation, not on its position on the plane.

In contrast, area scaling in homographies varies with position (e.g. under perspective a more distant square on the plane has a smaller image than one that is nearer)



The Projective Space \mathbb{P}^3

Points and Planes in \mathbb{P}^3

A point in \mathbb{P}^3 is defined as

$$X = [x; y; z; 1]$$

The incidence equation for a plane (i.e. $\mathbf{x} \in \pi$) is

$$ax + by + cz + d = 0$$

Which can be written in matrix form as

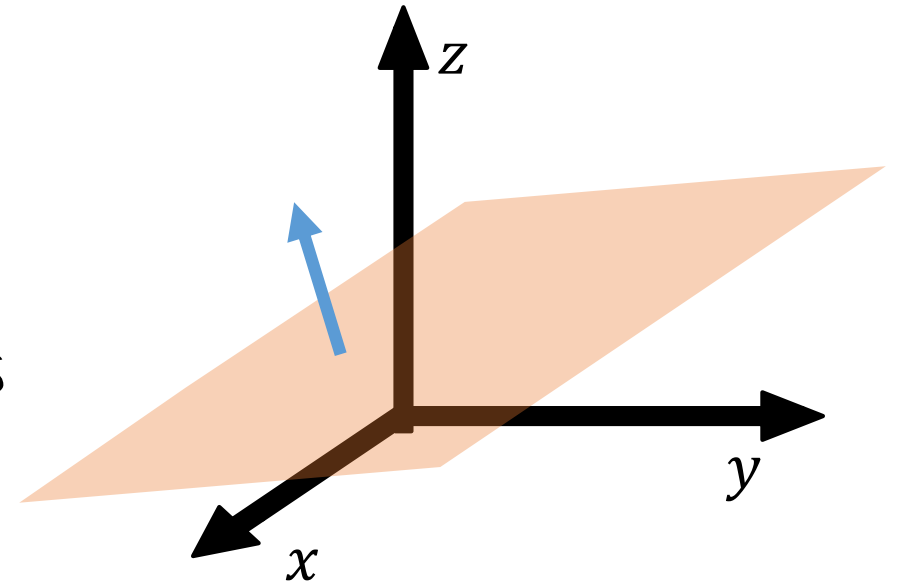
$$[a, b, c, d] \cdot X = 0$$

And this implies that the plane is identified by a vector in \mathbb{P}^3

$$\pi = [a; b; c; d]$$

Rmk planes have 3 degrees of freedom, since their equation holds up to a scaling of a parameter

Rmk planes in \mathbb{P}^3 are plays the same role as lines in \mathbb{P}^2



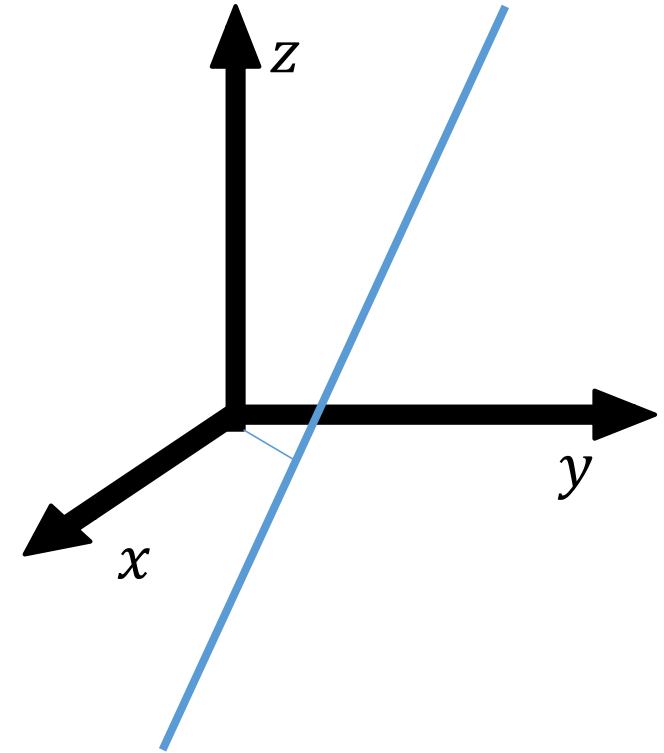
Lines in \mathbb{P}^3

More difficult to represent.

They have 4 degrees of freedom:

- Their direction
- The distance from the origin
(along the plane orthogonal to the direction)

Lines in \mathbb{P}^3 are defined as intersections of two planes



Number of parameters of a line

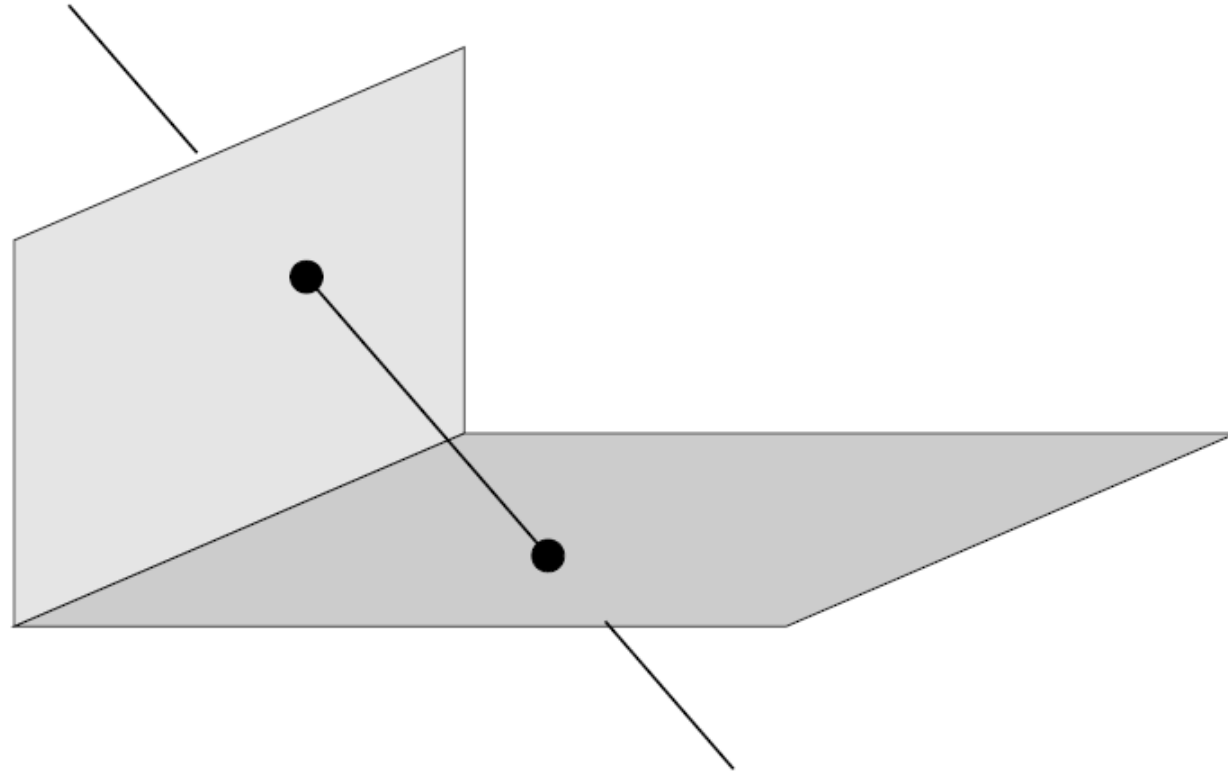


Fig. 3.1. *A line may be specified by its points of intersection with two orthogonal planes. Each intersection point has 2 degrees of freedom, which demonstrates that a line in \mathbb{P}^3 has a total of 4 degrees of freedom.*

The Plane at infinity

The plane at infinity has the canonical position

$$\boldsymbol{\pi}_\infty = [0; 0; 0; 1]$$

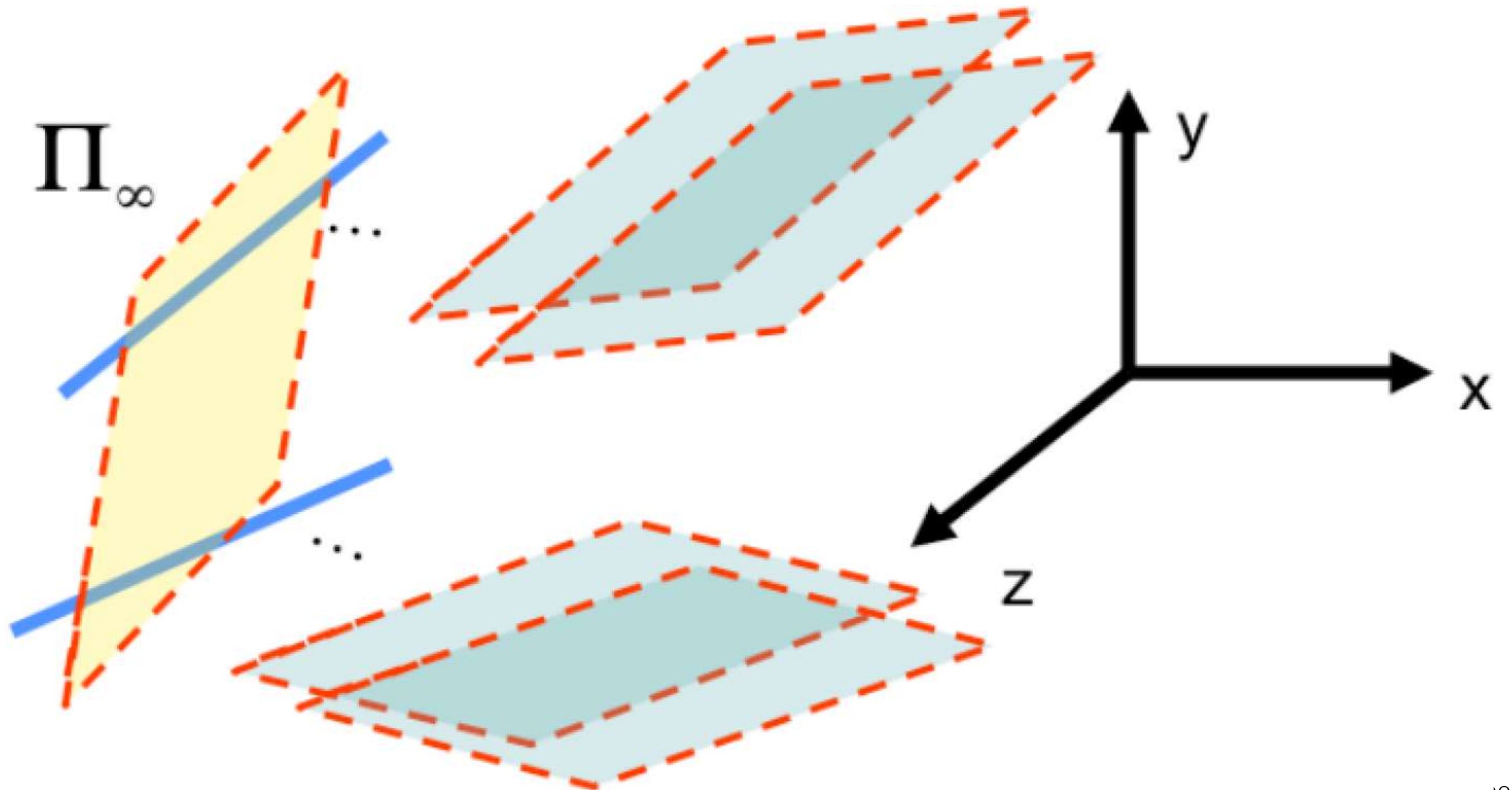
in affine 3D-space.

$\boldsymbol{\pi}_\infty$ contains ideal points (directions) $P_\infty = [x; y; z; 0]$, and enables the identification of affine properties such as parallelism.

In particular:

- Two planes are parallel if and only if their line of intersection is on $\boldsymbol{\pi}_\infty$
- A line is parallel to another line, or to a plane, if their point of intersection is on $\boldsymbol{\pi}_\infty$

The Plane at infinity

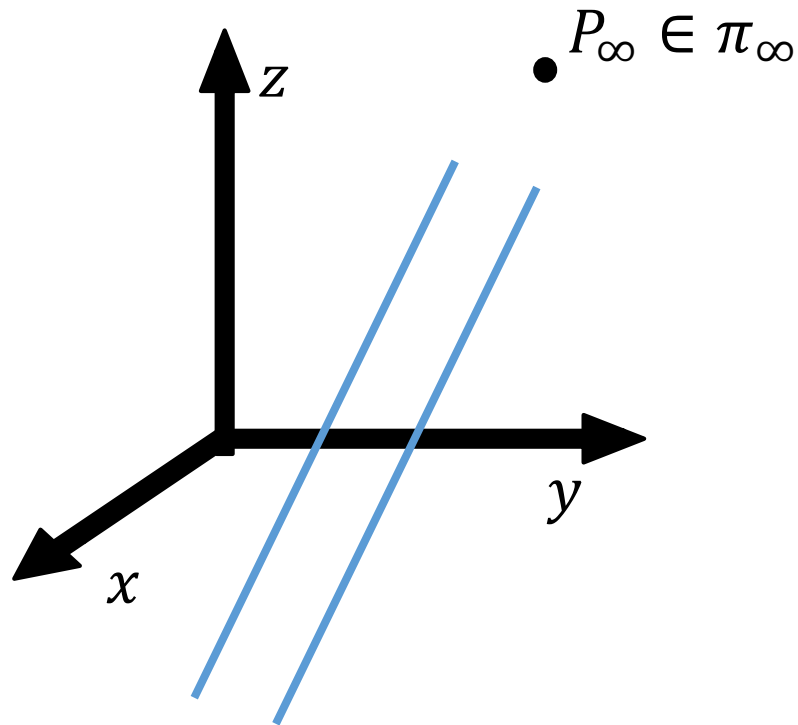


Ideal Points in \mathbb{P}^3

Ideal points in \mathbb{P}^3 are defined similarly to ideal points in \mathbb{P}^2

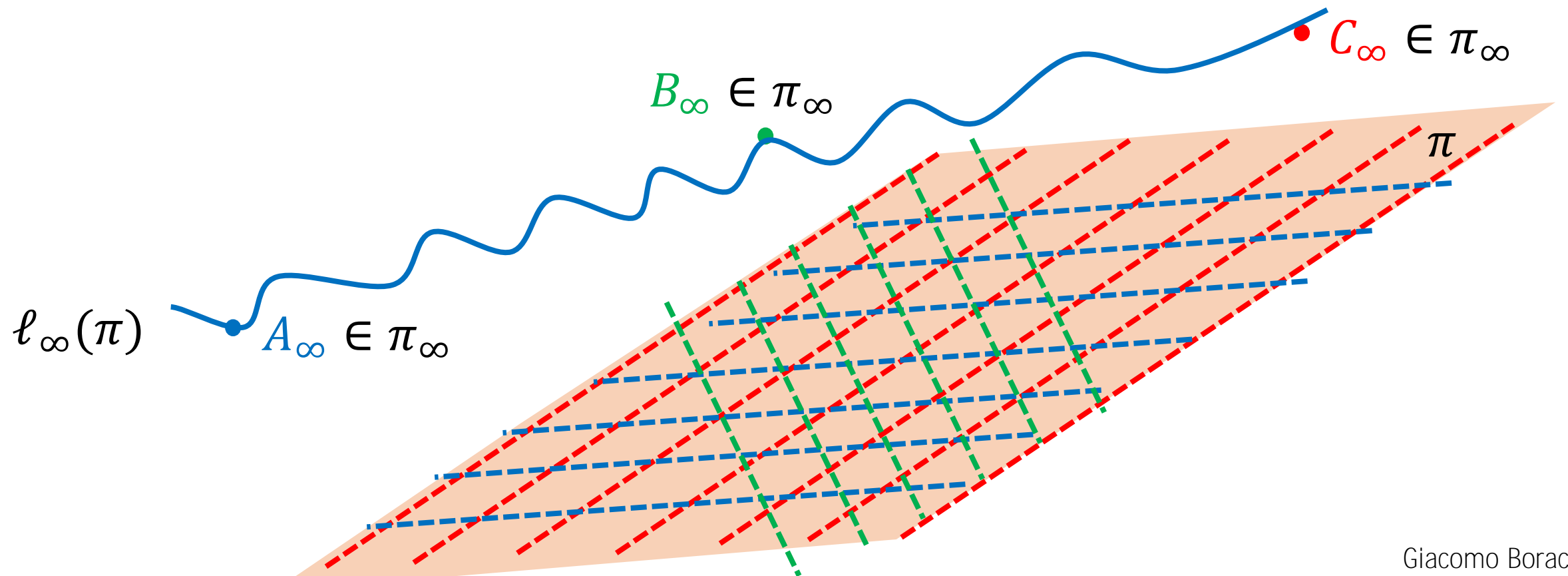
Ideal points are the intersection of parallel lines.

- All the parallel lines in the space intersect in the same ideal point P_∞

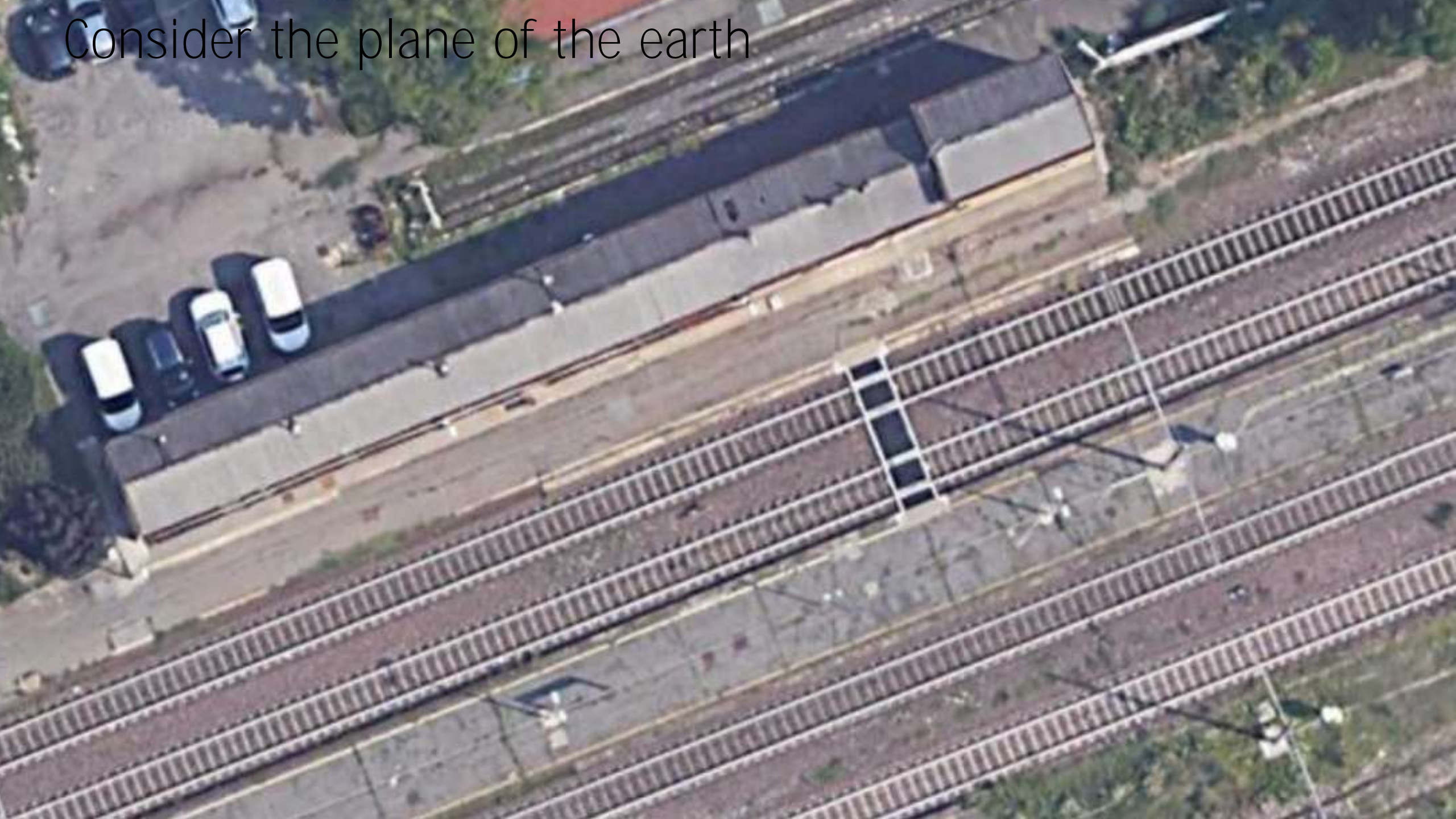


Planes in \mathbb{P}^3 and ideal lines

- A plane π contains infinite parallel lines along infinite directions
- Each set of parallel lines intersect in an ideal point (belonging to π_∞)
- All these ideal points lie on the ideal line $\ell_\infty(\pi)$: the line at infinity of π



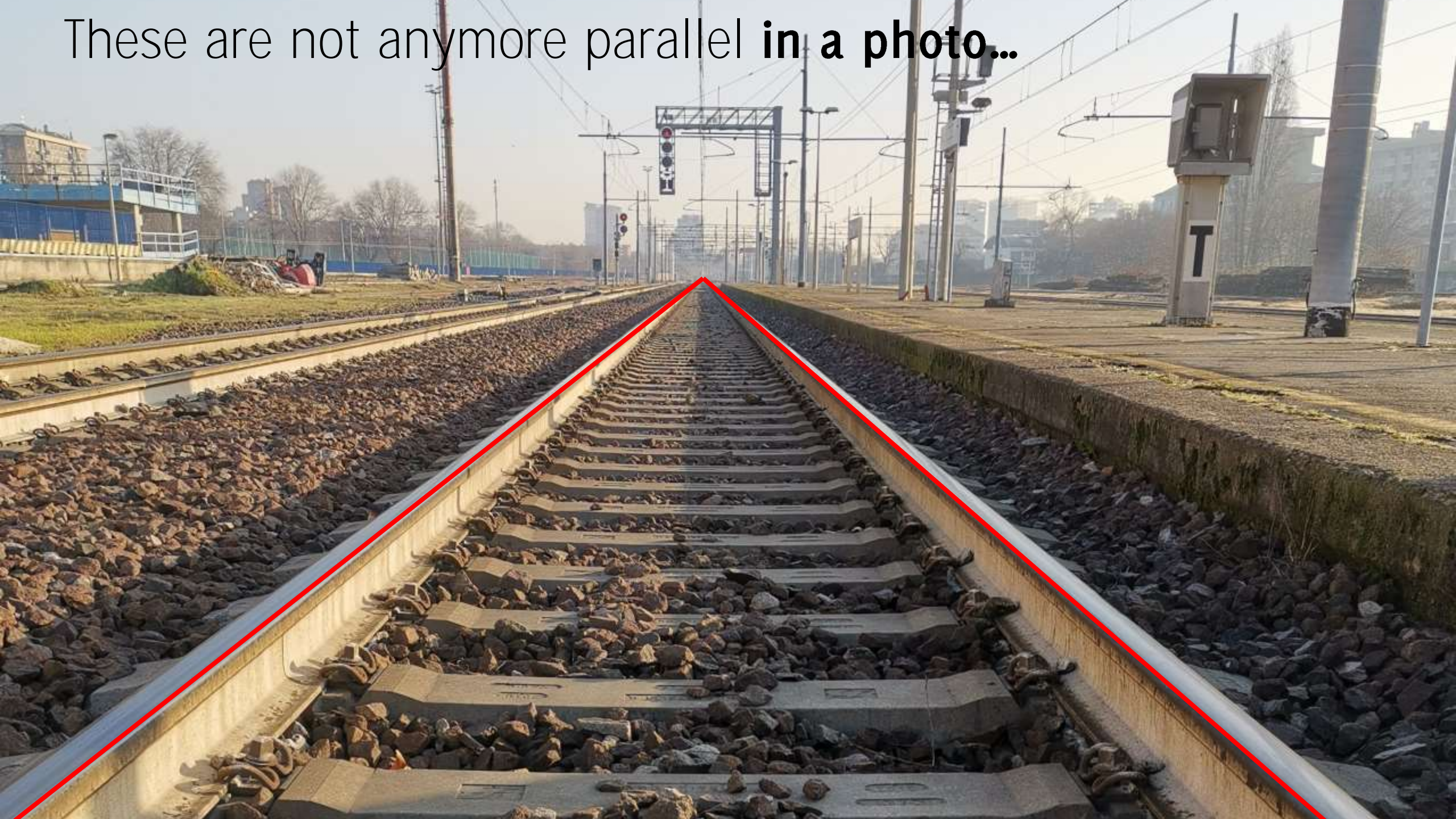
Consider the plane of the earth



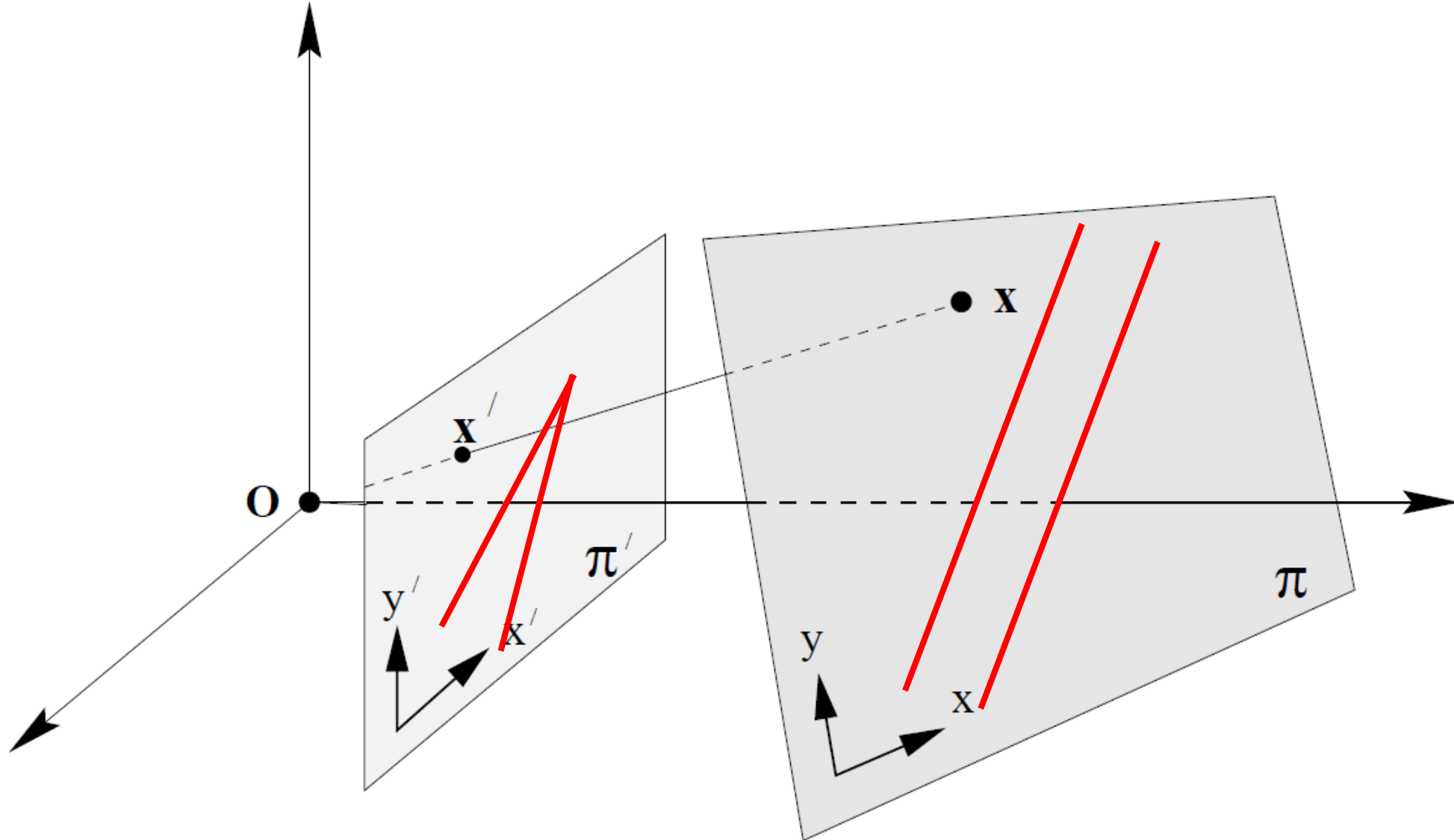
Consider the plane of the earth, which contains
a few parallel lines



These are not anymore parallel **in a photo...**



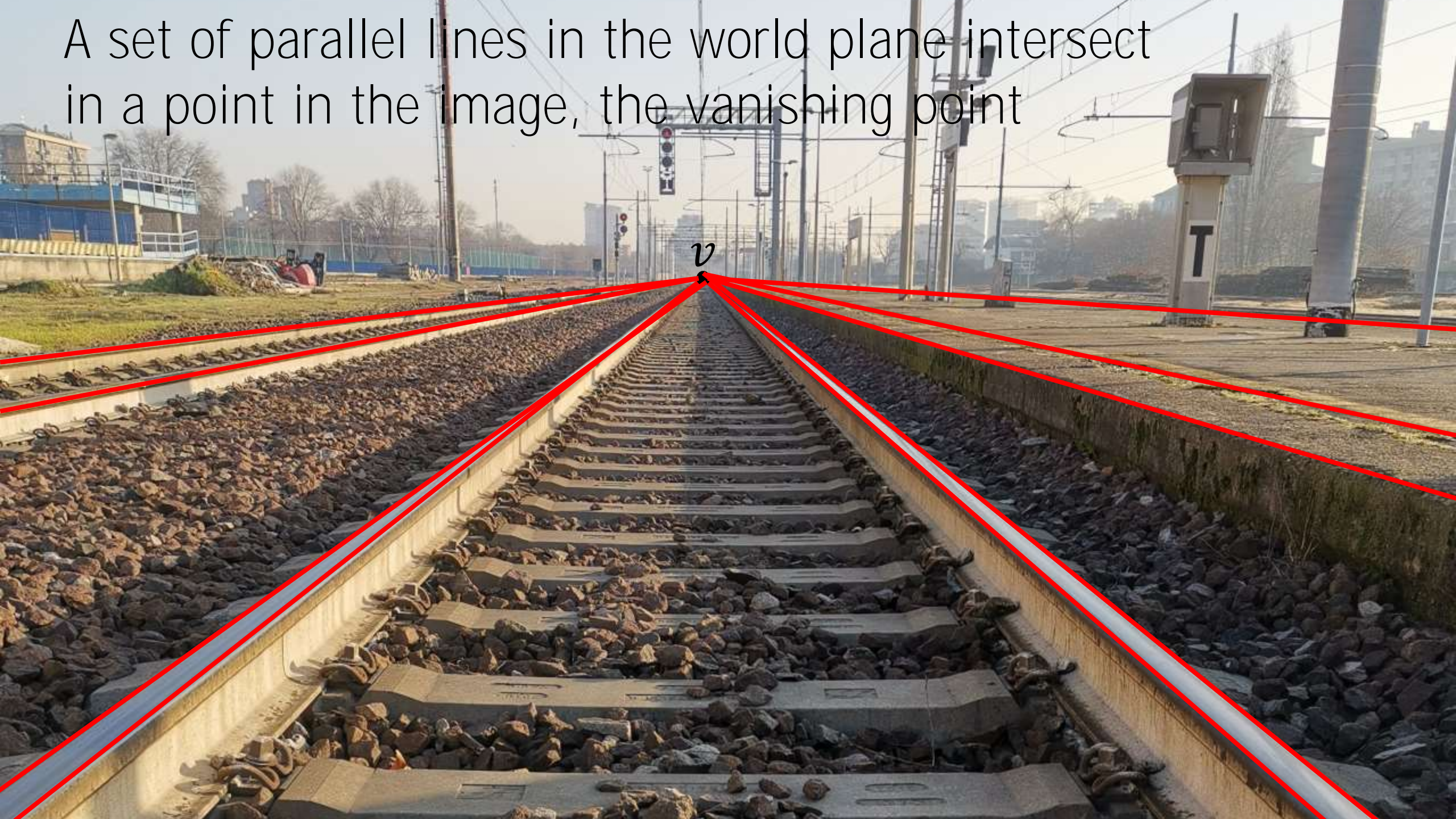
Homography is the mapping among two planes



Vanishing point



A set of parallel lines in the world plane intersect in a point in the image, the vanishing point



Vanishing points

Vanishing points are the projection in the image plane of ideal points in the 3D world

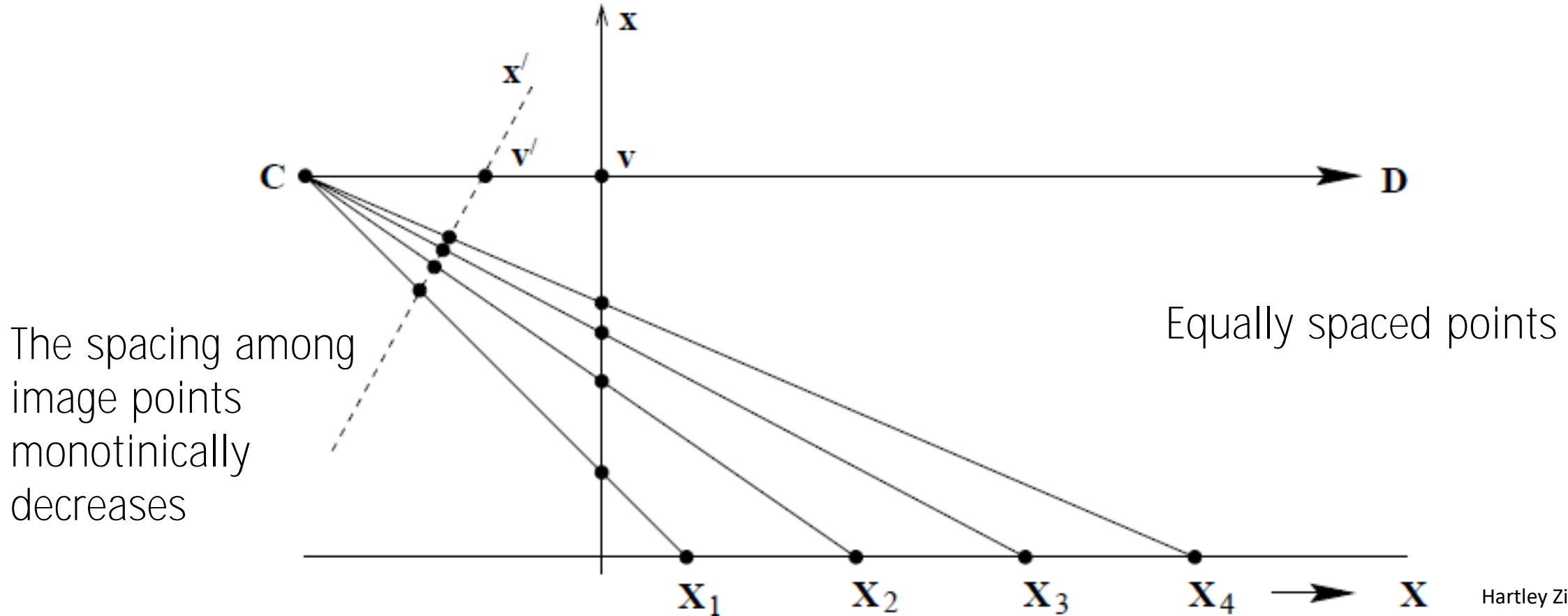
If a line has direction \mathbf{d} , then it intersects π_∞ in the point $\mathbf{X}_\infty = [\mathbf{d}; \mathbf{0}]$

Then the vanishing point \mathbf{v} is the image of \mathbf{X}_∞ by means of the camera matrix $\mathbf{M} = \mathbf{K}[\mathbf{I}_3, \mathbf{0}]$ (consider the camera center as the origin)

$$\mathbf{v} = \mathbf{M}\mathbf{X}_\infty = \mathbf{K}[\mathbf{I}_3, \mathbf{0}] \cdot [\mathbf{d}; \mathbf{0}] = \mathbf{K}\mathbf{d}$$

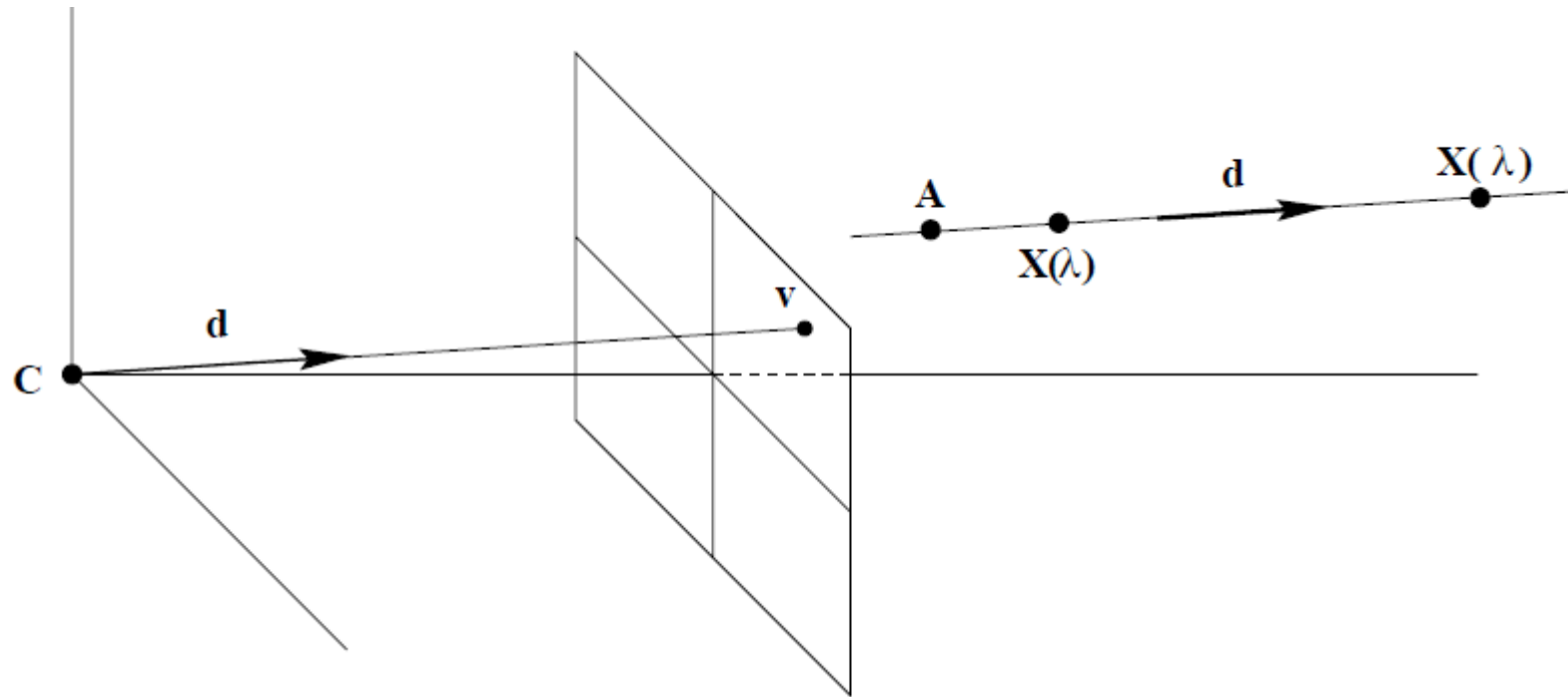
Vanishing points

When $X \rightarrow \infty$ along a line, its image converges to v , which is the intersection of the image plane with the line passing through C and directed as X



Vanishing points

Vanishing points are the projection in the image plane of ideal points in the 3D world



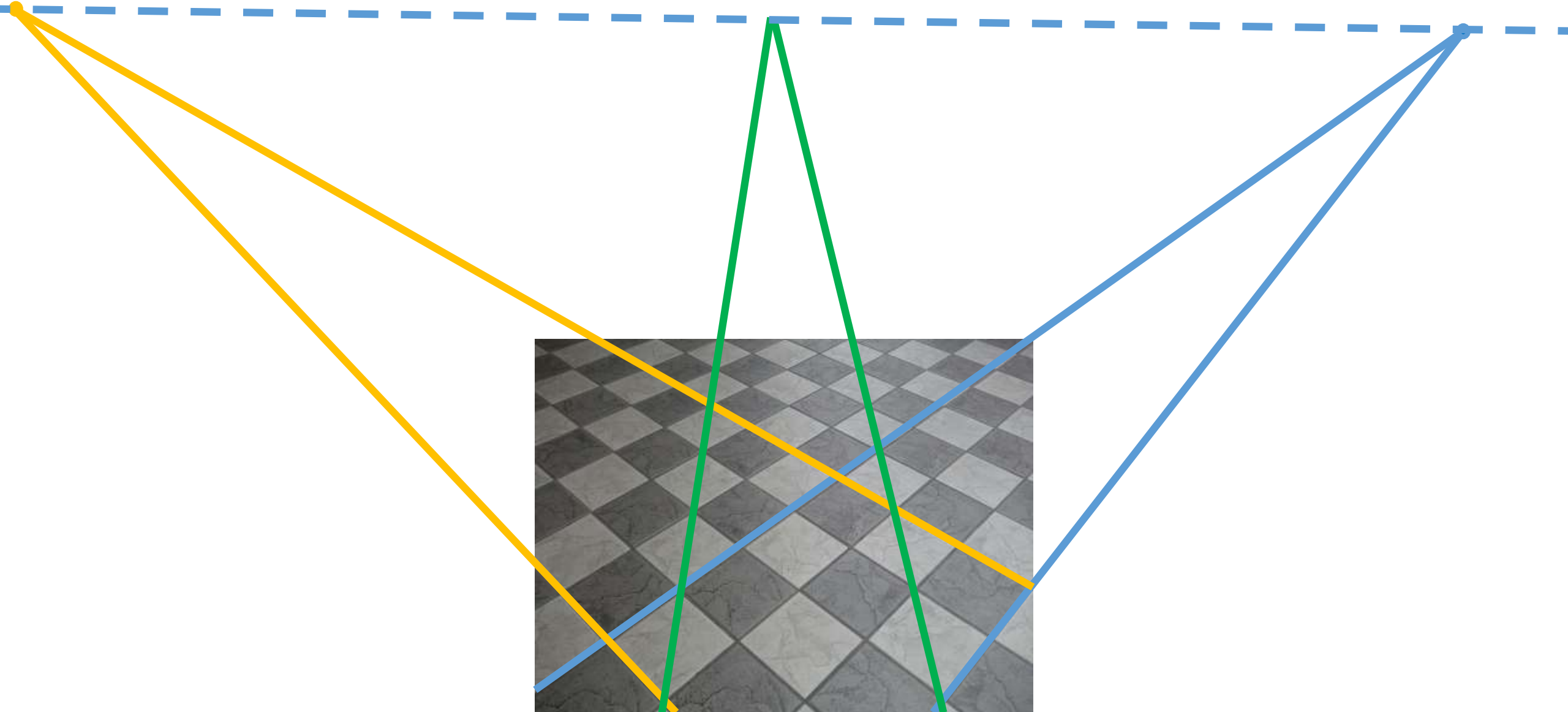
Horizon

Horizon or Vanishing line for a plane π is the image of the line at the infinity of that plane, $\ell_{\infty}(\pi)$

Horizon helps humans to intuitively deduce properties about the image that might not be apparent mathematically.

We can understand when two lines are parallel in the 3D world, since they intersect with the horizon

Horizon

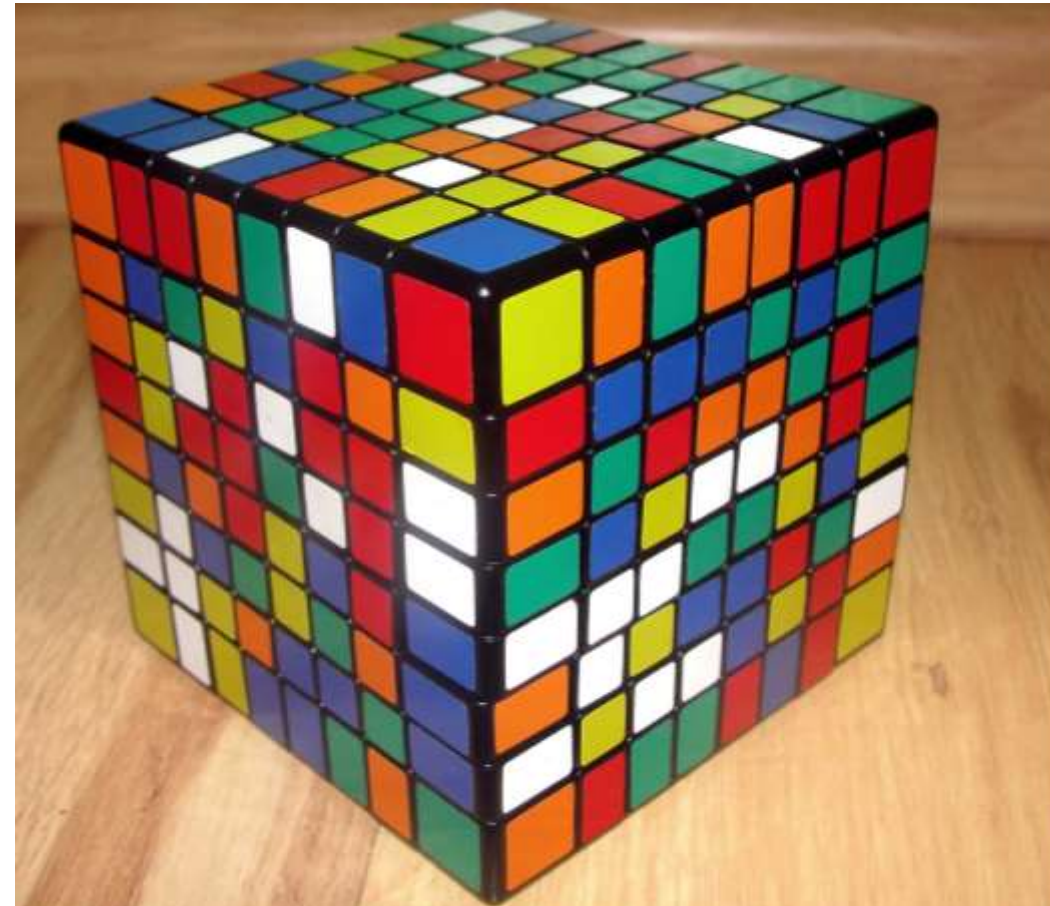


Remember!

The horizon line holds for a single plane of the 3D world

When the image contains multiple planes,
there is an horizon line for each of these

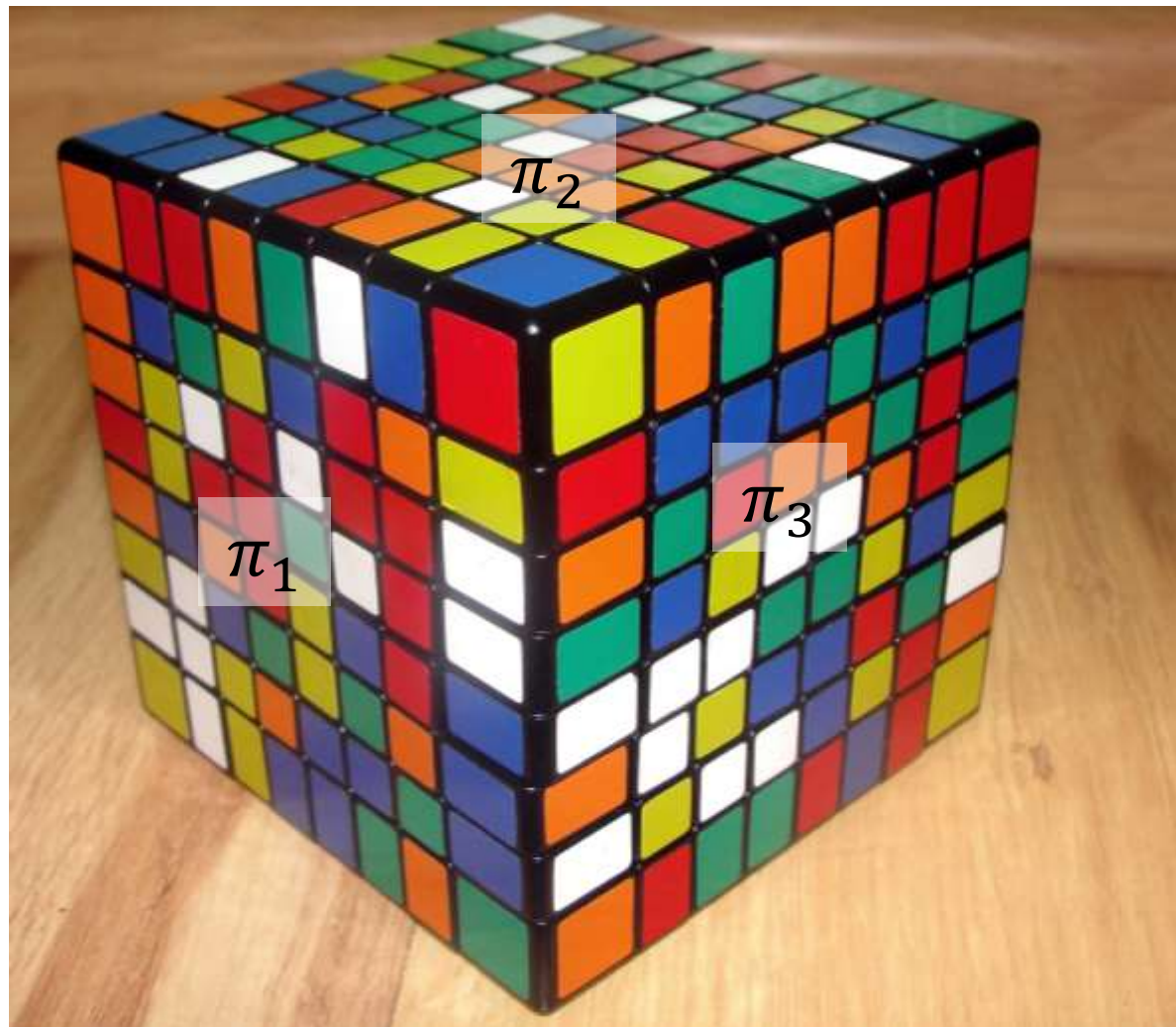
And ideal points of lines in a plane
are often not the same of lines on other
planes



$\ell_\infty(\pi_2)$

$\ell_\infty(\pi_3)$

$\ell_\infty(\pi_1)$



Affine Rectification

Idea:

- Identify the ℓ_∞ in the image
- transform the identified ℓ_∞ to its canonical position of $[0; 0; 1]$

Let $l = (l_1; l_2; l_3)$ be the image of the line at the infinity with $l_3 \neq 0$,

A suitable homography which maps l back to $\ell_\infty = [0; 0; 1]$ is

$$H = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Affine Rectification

Idea:

- Identify the ℓ_∞ in the image
- transform the identified ℓ_∞ to its canonical position of $[0; 0; 1]$

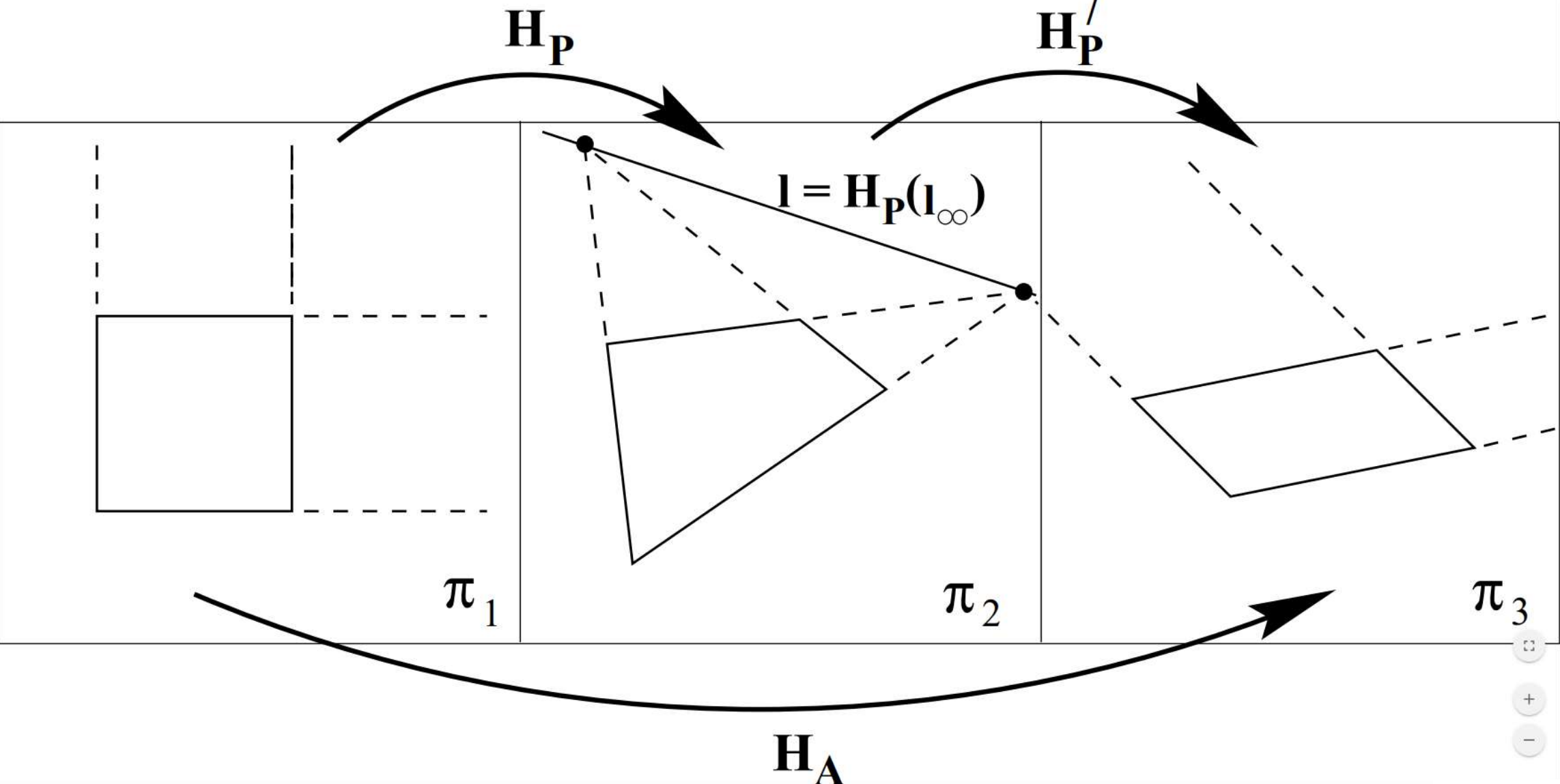
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A suitable homography which maps l back to $\ell_\infty = [0; 0; 1]$ is

$$H = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

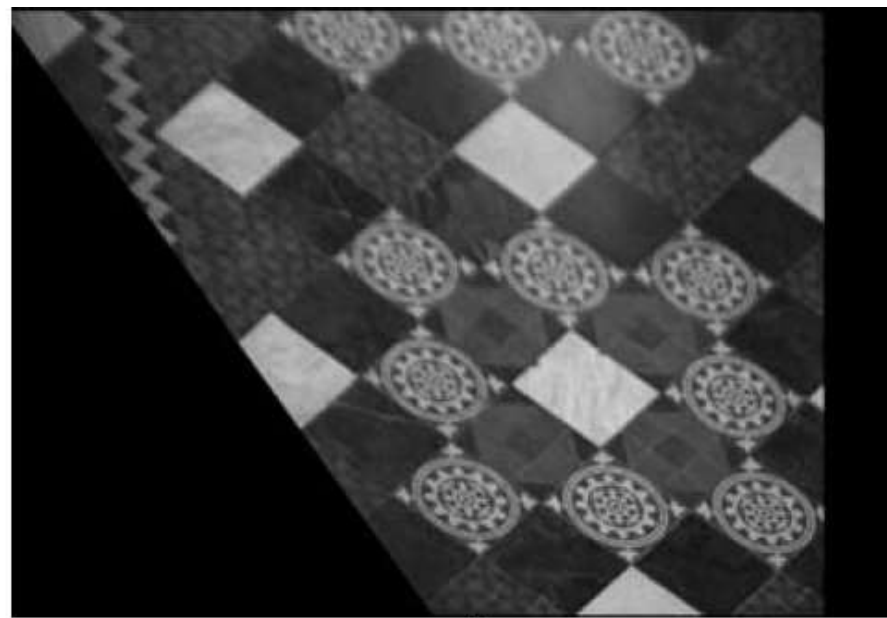
Rmk: l can be rescaled to improve conditioning of H

Affine Rectification

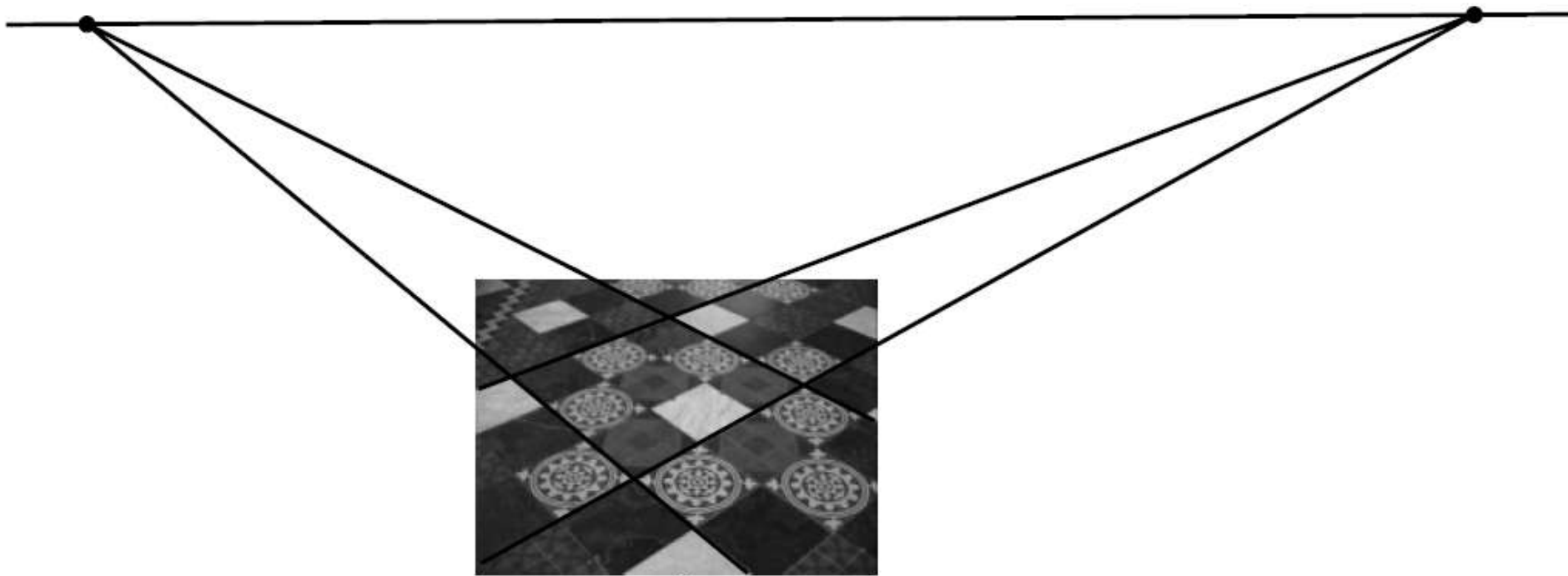




a



b



c

Single-View Geometry

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March 3rd, 2020

USI, Lugano

Book: HZ, chapter 2,3

Outline

- DLT Algorithm for Least Squares Fitting in \mathbb{P}^2
 - Homography estimation
 - Camera Calibration
- Applying Point Transformations to Images
- Conics in \mathbb{P}^2
- Conic Fitting

DLT algorithm

How to estimate homographies and much more...

HZ chapter 4

DLT

Direct Linear Transformation (DLT) algorithm, solves many relevant problems in Computer Vision

- 2D Homography estimation
- Camera projection matrix estimation (projections from 3D to 2D)
- Fundamental matrix computation
- Trifocal tensor computation

Homography estimation

We consider a set of point correspondences

$$\{(\mathbf{x}'_i, \mathbf{x}_i), i = 1, \dots, 4\}$$

belonging to two different images. Our problem is to compute a $H \in \mathbb{R}^{3 \times 3}$ such that $\mathbf{x}'_i = H\mathbf{x}_i$ for each i .

Rmk if we look at \mathbf{x} as a 3d vector the equality $\mathbf{x}'_i = H\mathbf{x}_i$ does not hold
Equality holds in \mathbb{P}^2 (but linear systems are solved as in \mathbb{R}^3).

What makes the DLT is distinct from standard cases since the left and right sides of the defining equation can differ by an unknown multiplicative factor which is dependent on the number of equations.

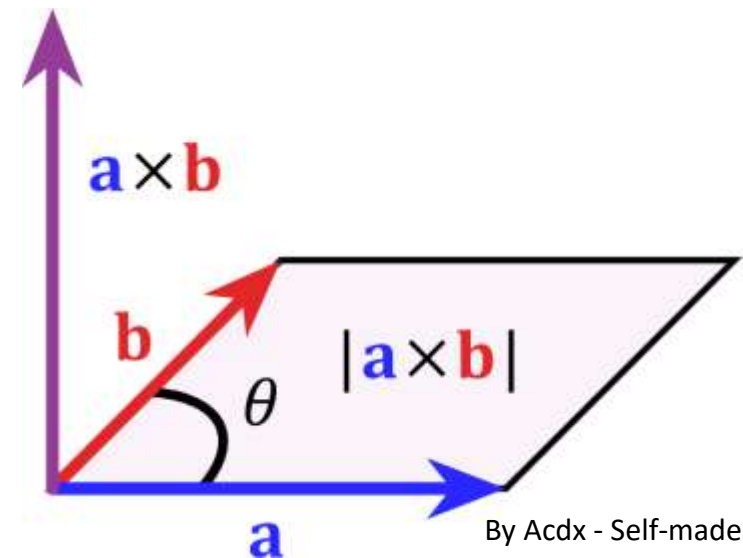
DLT for Homography Estimation

In \mathbb{R}^3 , \mathbf{x}'_i and \mathbf{x}_i have the same direction, but may differ in magnitude.

Therefore collinearity constraints can be written as

$$\mathbf{x}'_i \times H\mathbf{x}_i = \mathbf{0}, i = 1, \dots, 4$$

where $\mathbf{0} = [0; 0; 0]$



DLT for Homography Estimation

Let $\mathbf{x}_i = [x_i; y_i; w_i]$ and $H = [\mathbf{h}_1^\top; \mathbf{h}_2^\top; \mathbf{h}_3^\top]$ being \mathbf{h}_1^\top the first row...

Then the cross product can be written as..

$$\mathbf{x}_i' \times H\mathbf{x}_i = \begin{pmatrix} y_i' \mathbf{h}_3^\top \mathbf{x}_i - w_i' \mathbf{h}_2^\top \mathbf{x}_i \\ w_i' \mathbf{h}_1^\top \mathbf{x}_i - x_i' \mathbf{h}_3^\top \mathbf{x}_i \\ x_i' \mathbf{h}_2^\top \mathbf{x}_i - y_i' \mathbf{h}_1^\top \mathbf{x}_i \end{pmatrix}$$

DLT for Homography Estimation

And after some linear algebra, this can be expressed in a matrix form as

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i \mathbf{x}_i^\top & y'_i \mathbf{x}_i^\top \\ w'_i \mathbf{x}_i^\top & \mathbf{0}^\top & -x_i \mathbf{x}_i^\top \\ -y'_i \mathbf{x}_i^\top & x'_i \mathbf{x}_i^\top & \mathbf{0}^\top \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix} = 0$$

This is a 3×9 matrix multiplied times a vector of 9 elements

Rmk The equation is an equation *linear* in the unknown vector

$$\mathbf{h} = [\mathbf{h}_1 ; \mathbf{h}_2 ; \mathbf{h}_3]$$

Rmk the three rows of the matrix are linearly dependent (the third row is the sum of $-x_i/w_i$ times the first row and $-y_i/w_i$ times the second)

DLT for Homography Estimation

Thus, keep only two rows giving rise to an homogeneous linear system

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i \mathbf{x}_i^\top & y'_i \mathbf{x}_i^\top \\ w'_i \mathbf{x}_i^\top & \mathbf{0}^\top & -x_i \mathbf{x}_i^\top \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix} = \mathbf{0}$$

Which can be written as

$$A_i \mathbf{h} = \mathbf{0}, \quad i = 1, \dots, 4$$

Rmk remember that \mathbf{h} has been unrolled row-wise, not column-wise

DLT for Homography Estimation

Stacking 4 point correspondences gives an 8×9 matrix

$$A\mathbf{h} = \mathbf{0}$$

We are not interested in the trivial solution $\mathbf{h} = \mathbf{0}$

Solve it as $\mathbf{h} = RNS(A)$ and arbitrarily imposing $\|\mathbf{h}\| = 1$

DLT in the overdetermined case

Let us assume $n > 4$ point correspondences are given, such that A is a $2n \times 9$ matrix

$$A\mathbf{h} = \mathbf{0}$$

The system is overdetermined. We are not interested

- in the trivial solution $\mathbf{h} = \mathbf{0}$ and
- not even in an exact solution, since typically this does not exist because of noise in the measurements $\mathbf{x}'_i, \mathbf{x}_i$

Thus, impose the constraint $\|\mathbf{h}\| = 1$ and minimize a cost function

$$\mathbf{h}^* = \operatorname{argmin} \|A\mathbf{h}\|_2 \text{ s.t. } \|\mathbf{h}\|_2 = 1$$

DLT in the overdetermined case

The solution of this problem is obtained by

$$\mathbf{h}^* = \operatorname{argmin} \|\mathbf{A}\mathbf{h}\|_2 \text{ s. t. } \|\mathbf{h}\|_2 = 1$$

The solution is the (unit) eigenvector of $\mathbf{A}^T \mathbf{A}$ with least eigenvalue.

Equivalently, the solution is the unit singular vector corresponding to the smallest singular value of \mathbf{A} . (See HZ A5.3(p592))

Specifically, if $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ with \mathbf{D} diagonal with positive diagonal entries, arranged in descending order down the diagonal, then \mathbf{h} is the last column of \mathbf{V} .

Rmk DLT algorithm minimizes the residual $\|\mathbf{A}\mathbf{h}\|$, which has to be interpreted as an *algebraic error*

DLT and the reference system

Are the outcome of DLT independent of the reference system being used to express \mathbf{x}' and \mathbf{x} ?

Unfortunately DLT is not invariant to similarity transformations.

Therefore, it is necessary to apply a normalizing transformation to the data before applying the DLT algorithm.

Normalizing the data makes the DLT invariant to the reference system, as it is always being estimated in a canonical reference

Normalization is also called Pre-conditioning

Preconditioning

Needed because in homogeneous coordinate systems, components typically have very different ranges

- Row and Column indexes ranges in $[0 - 4K]$
- Third component is **1**

Define a mapping

$$x \rightarrow Tx$$

That brings all the points «around the origin» and rescale each component to the same range (say at an average distance $\sqrt{2}$ from the origin)

Preconditioning

$$\mathbf{x} \rightarrow T\mathbf{x}$$

It's a similarity scaling in \mathbb{P}^2 (with $\theta = 0$ since we do not need rotations)

$$T = \begin{bmatrix} 1/s & 0 & -t_x/s \\ 0 & 1/s & -t_y/s \\ 0 & 0 & 1 \end{bmatrix}$$

The preconditioning for a set of points X of \mathbb{P}^2 is defined as

$$t_x = \text{mean}(X(1, :))$$

$$t_y = \text{mean}(X(2, :))$$

Which brings the barycentre of X to the origin, the scaling is

$$s = \frac{\text{mean}(\text{std}(X, 2))}{\sqrt{2}}$$

Homography estimation with preconditioning

Estimate the homography between two sets of points X, X'

1. Compute T, T' preconditioning transformation of X, X'
2. Apply transformation

$$X_c = TX, \quad X'_c = T'X'$$

1. Estimate the homography from X_c and X'_c ,

$$H = DLT(X_c, X'_c)$$

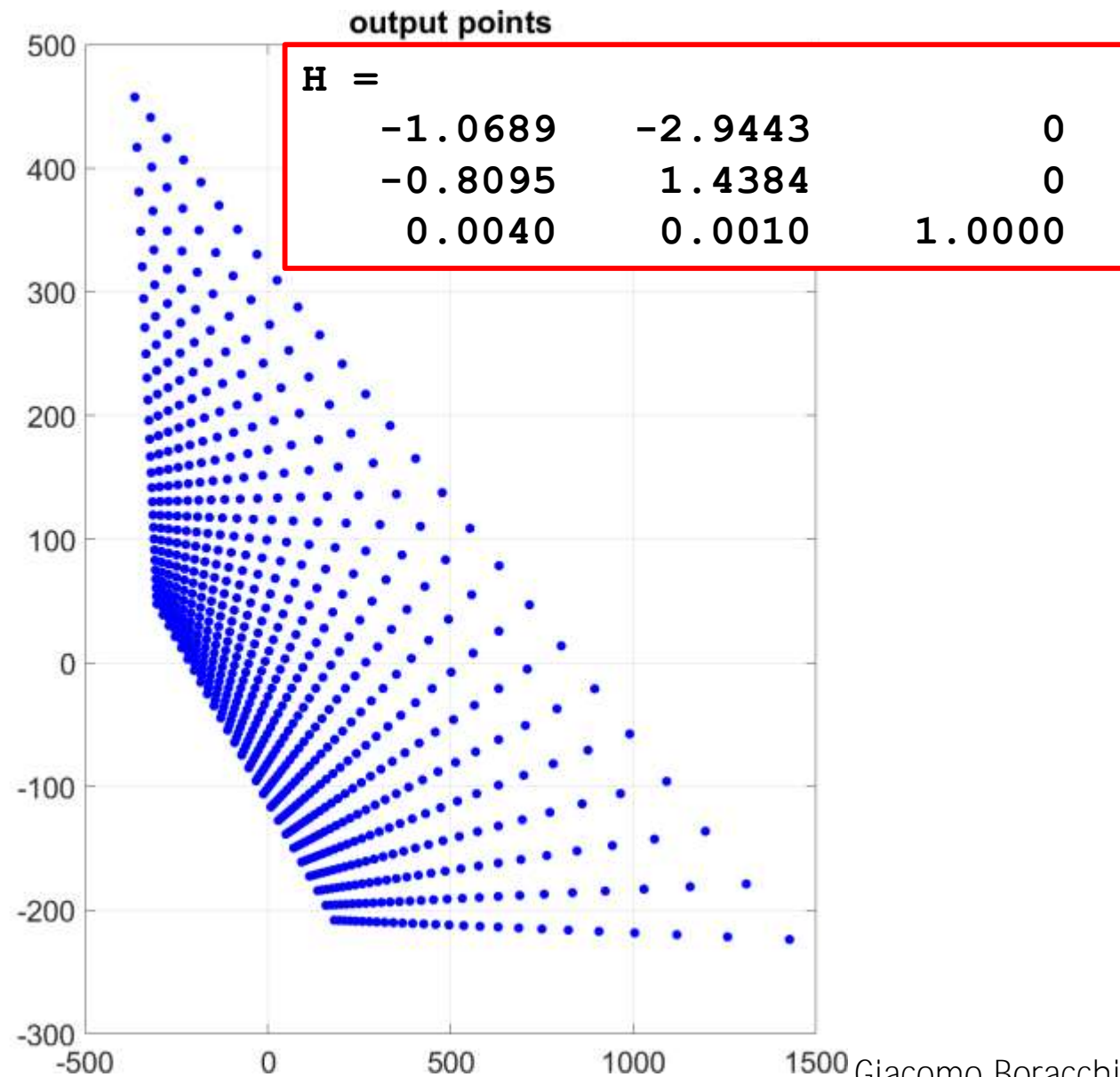
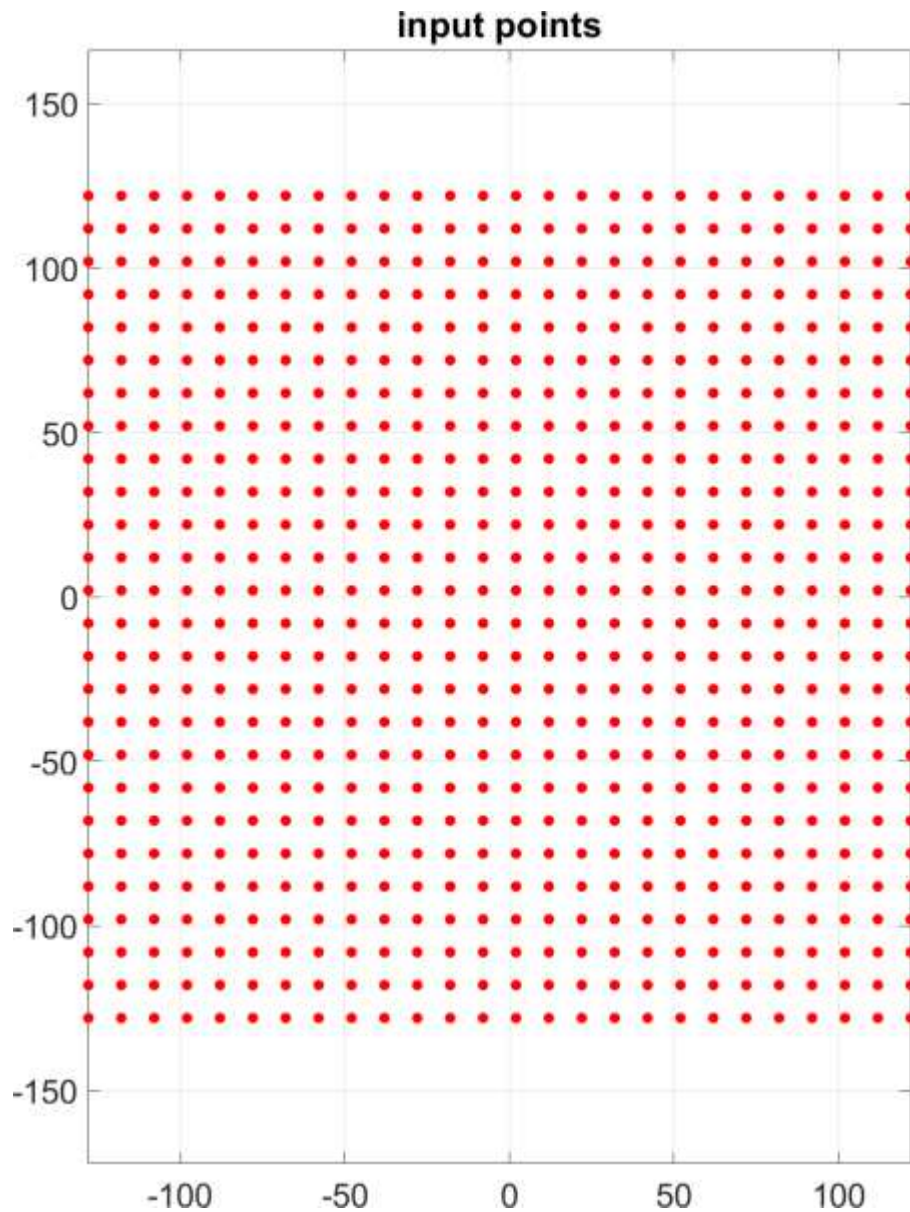
1. Define the transformation

$$H_c = T'^{-1}HT$$

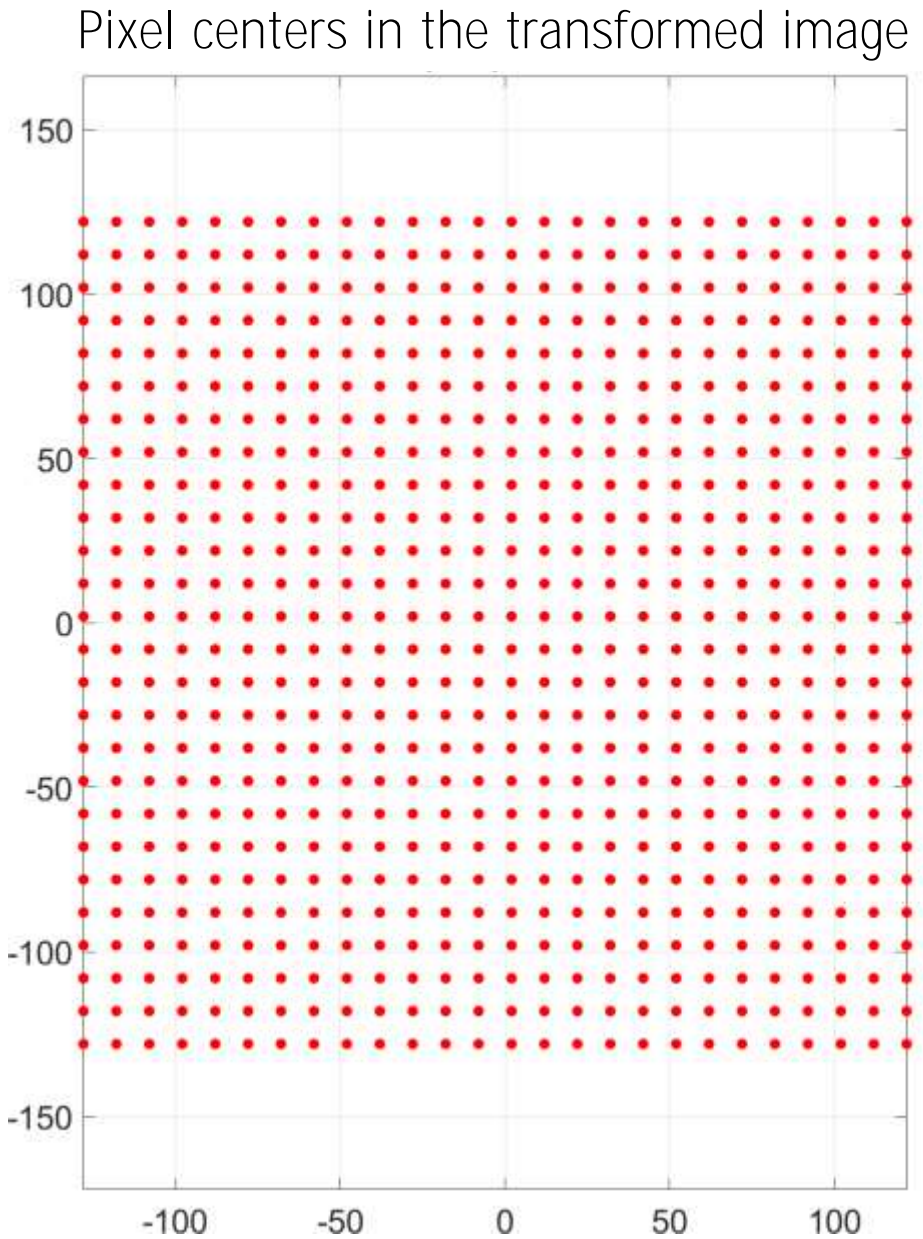
How to apply linear transformations to an image?

...these are transformations between points, not pixel intensities

Apply the inverse of the desired transformation on a grid covering the output image



Here is the grid of the transformed image

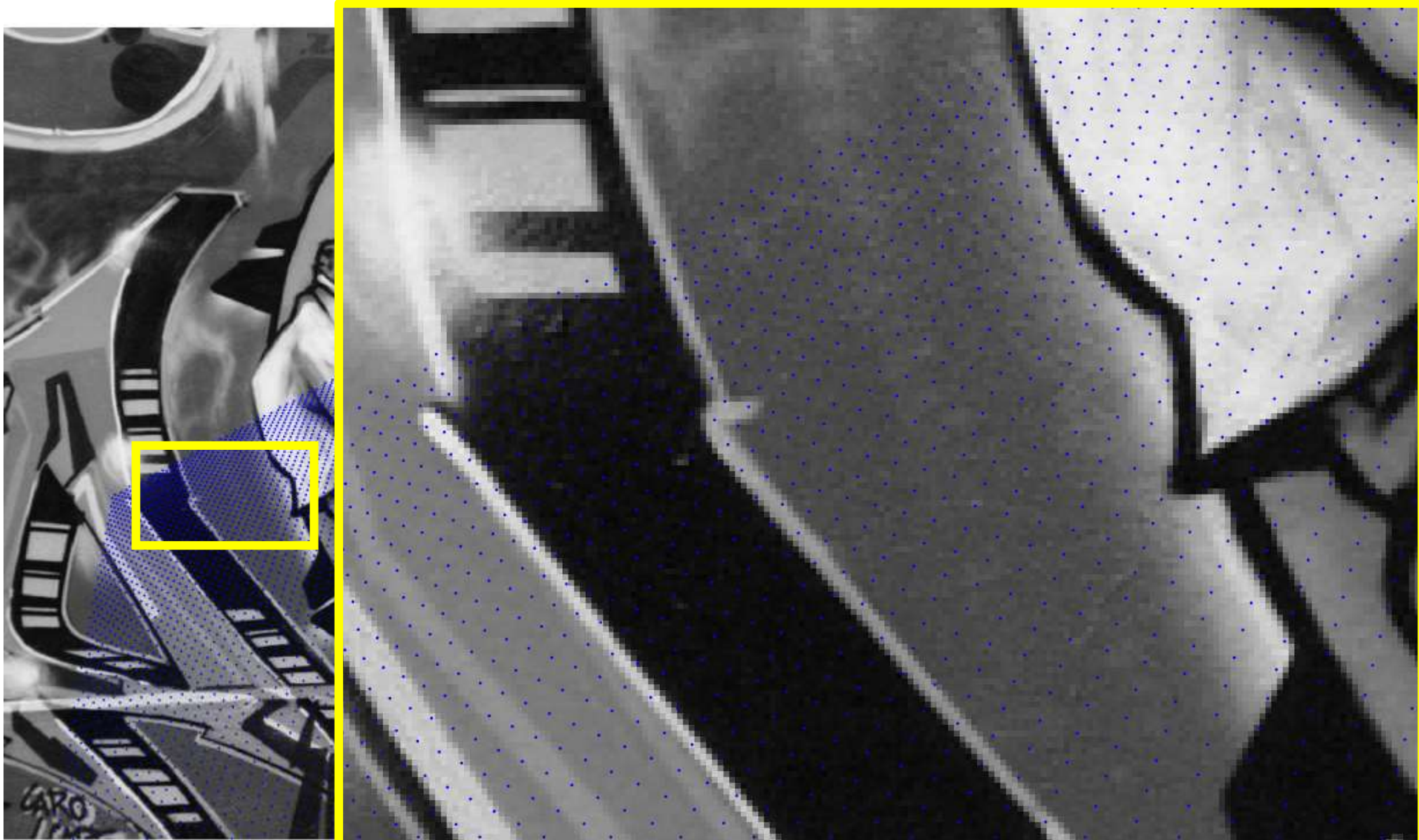


Place the grid transformed by H^{-1} on the input image

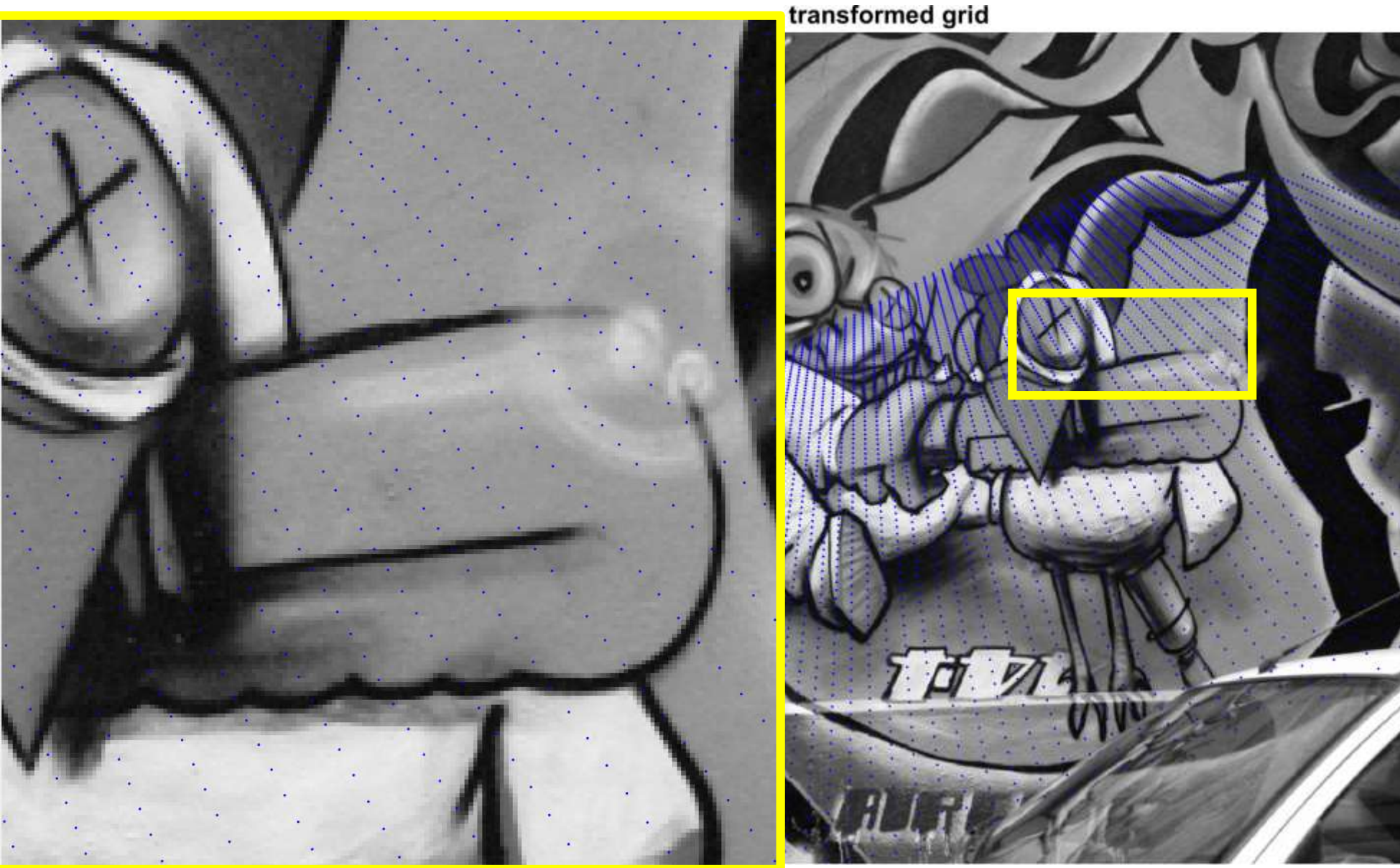
transformed grid



Place the grid transformed by H^{-1} on the image



Place the grid transformed by H^{-1} on the image



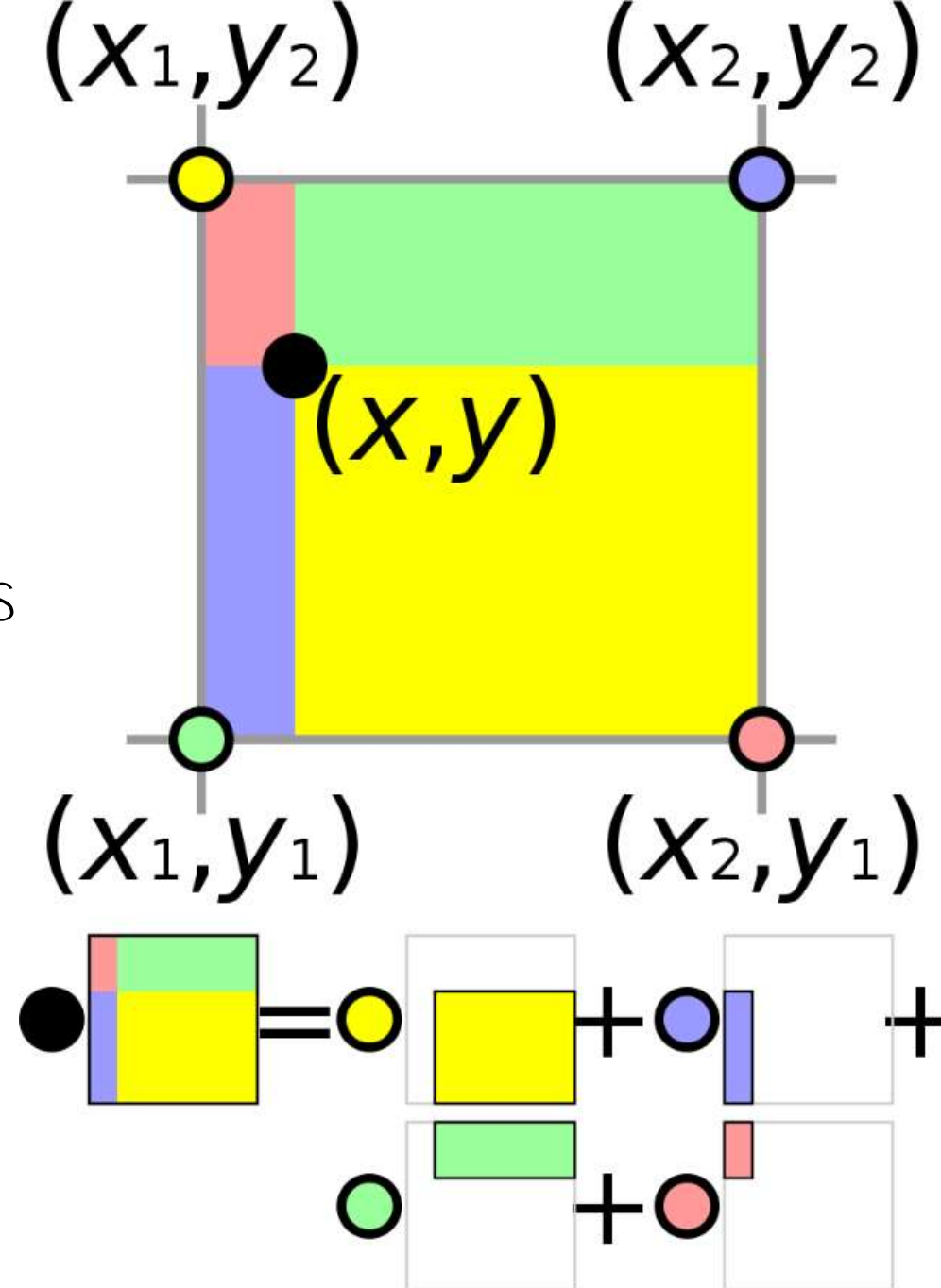
Define image intensity at blue points by means of bilinear interpolation

transformed grid



Bilinear Interpolation

Intensity value over the transformed grid (x, y) are defined by interpolating values of neighbouring pixels $\{(x_i, y_i), i = 1, 2\}$. Several interpolation options are viable. Bilinear interpolation is a weighted average with weights proportional to the areas illustrated here.



Bilinear Interpolation

Let (x, y) be the point and $f(x, y)$ the value we want to estimate.

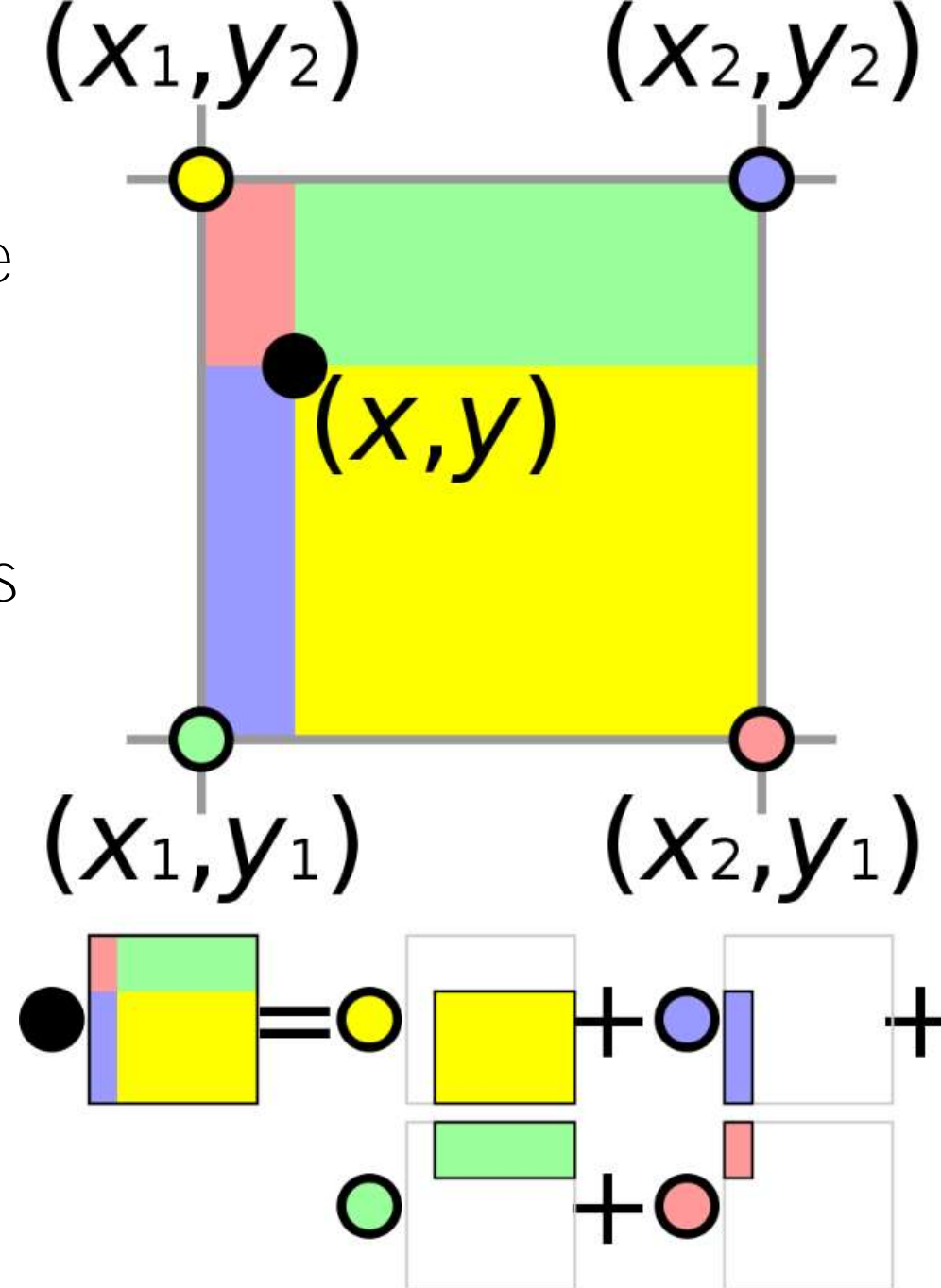
Bilinear interpolation corresponds to

- Perform a 1D-linear interpolation along x axis to estimate $f(x, y_1)$ and $f(x, y_2)$

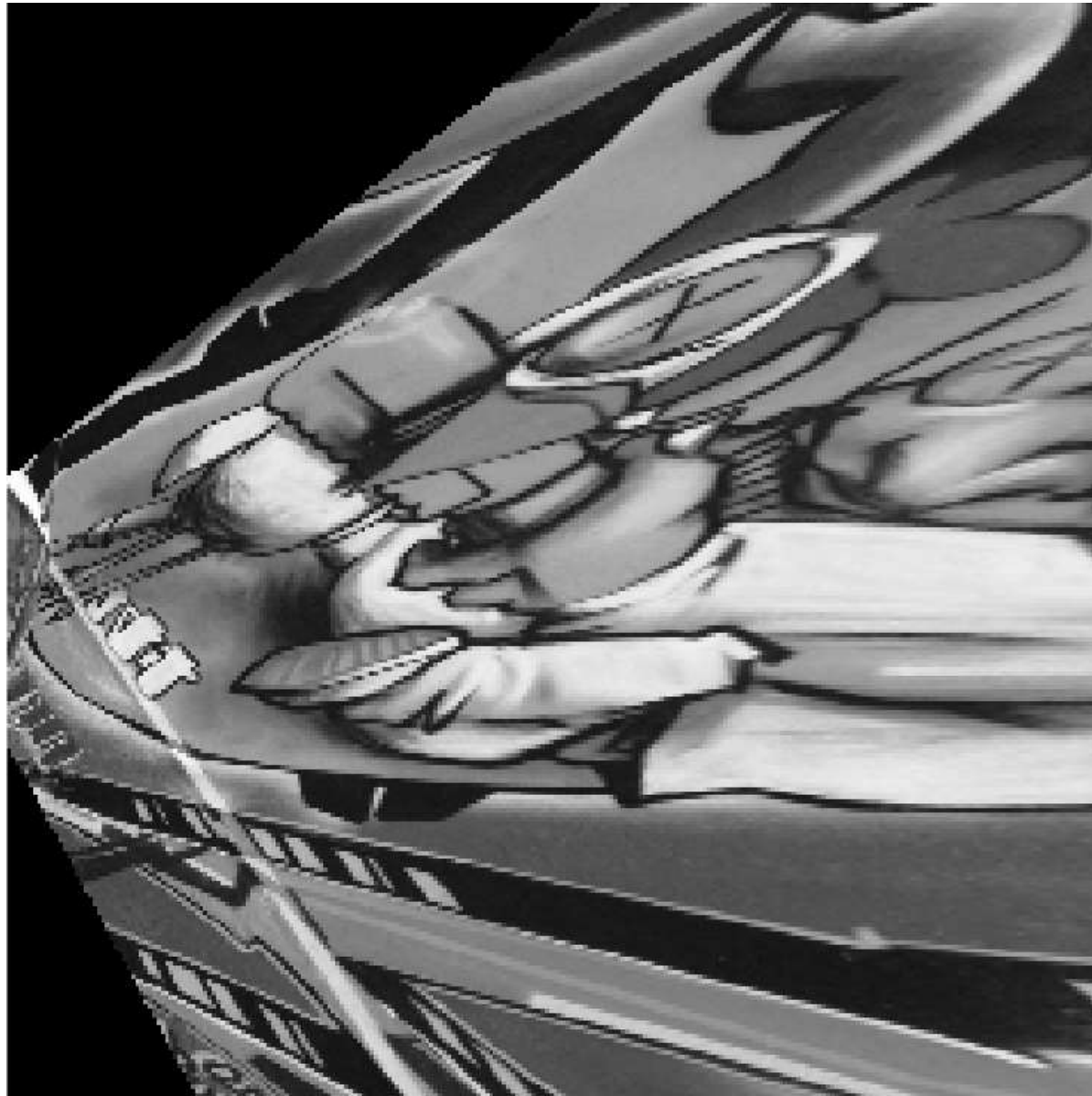
$$f(x, y_1) = \frac{x_2 - x}{x_2 - x_1} f(x_1, y_1) + \frac{x - x_1}{x_2 - x_1} f(x_2, y_1)$$

$$f(x, y_2) = \frac{x_2 - x}{x_2 - x_1} f(x_1, y_2) + \frac{x - x_1}{x_2 - x_1} f(x_2, y_2)$$

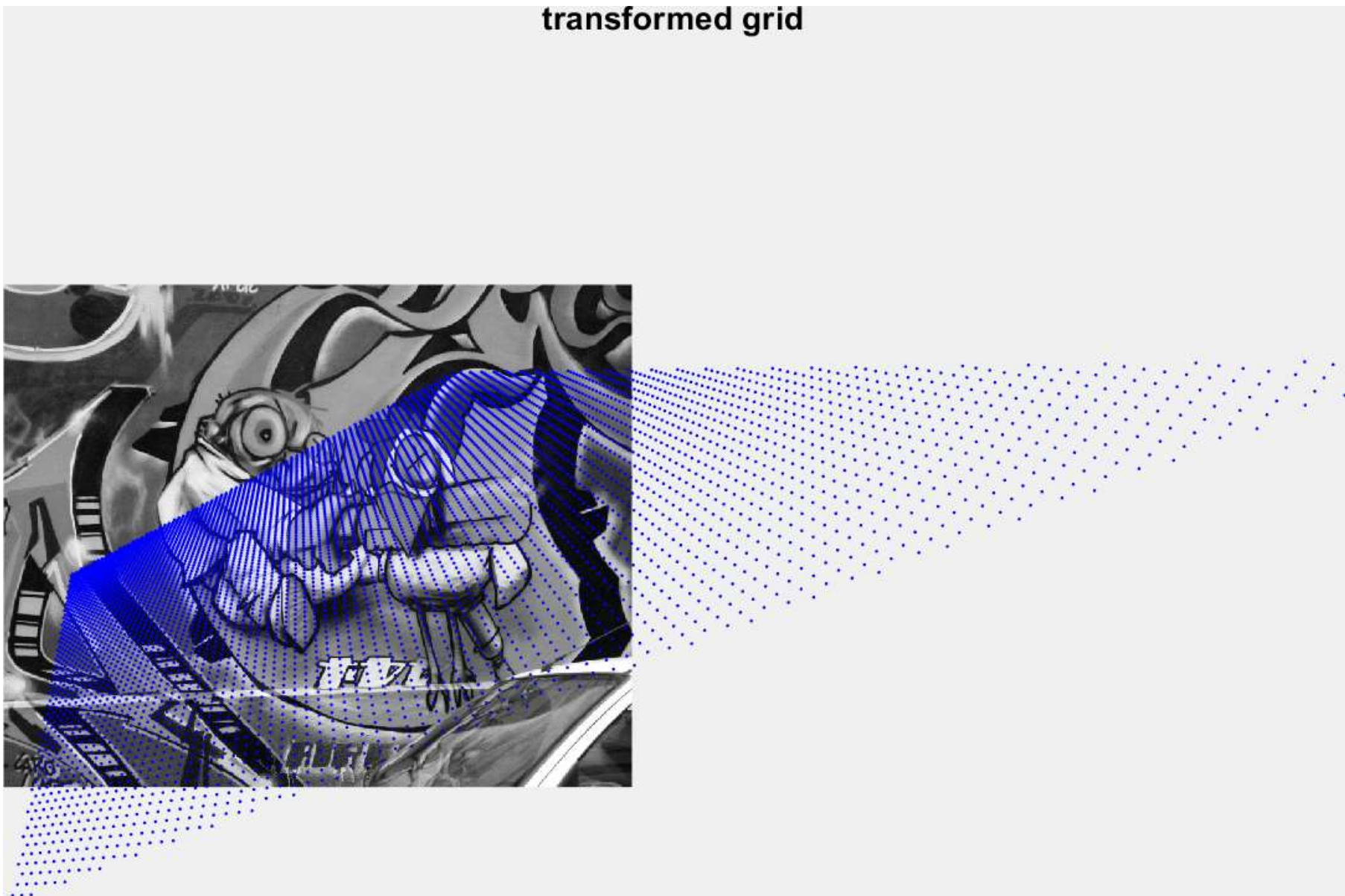
- Perform 1D-linear interpolation along y axis but considering $f(x, y_1)$ and $f(x, y_2)$



Re-arrange the new estimated values over a regular grid to obtain the transformed image



Black pixels are points out of the image grid



Single-View Geometry

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March 10th, 2020

USI, Lugano

Book: HZ, chapter 2

Outline

- Implications of image calibration
 - Camera Center
 - Viewing Rays
 - Factorization of M
- Conics and Conic Fitting
 - Comments on Error Minimization
- Measuring Angles
- Single View Rectification

The Camera Center \mathcal{C}

The 3D coordinates of the center of a camera having matrix $M \in \mathbb{R}^{3,4}$ satisfy

$$\mathcal{C} \in RNS(M)$$

Where $RNS(\cdot)$ denotes the Right Null Space.

Note that when the $RNS(M)$ has dimension 1 (i.e. always but in degenerate cases) all the points $\mathcal{C} \in RNS(M)$ coincide in the homogenous space.

The Camera Center \mathcal{C}

Proof

Let us consider $\mathcal{C} \in RNS(M)$, then

$$MP = M(P + \lambda\mathcal{C}) \quad \forall \lambda \in \mathbb{R}, \quad \forall P \in \mathbb{P}^3$$

This means that the line $P + \lambda\mathcal{C}$ does not change its projection through M , thus $P + \lambda\mathcal{C}$ is a viewing ray. Since this has to hold $\forall P \in \mathbb{P}^3$, this means that \mathcal{C} is the camera center.

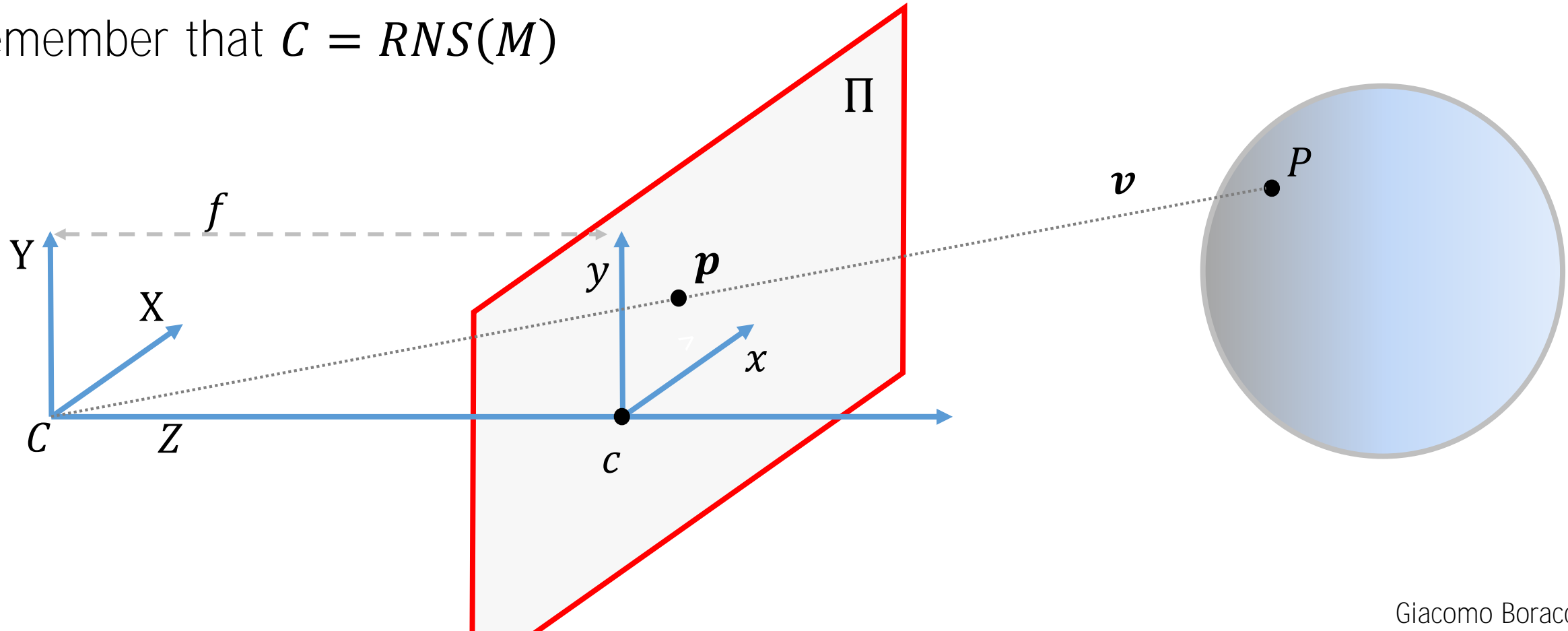
The converse is trivial because given the camera center \mathcal{C} , then $M(P + \lambda\mathcal{C}) = MP \quad \forall P \in \mathbb{P}^3$ (since $P + \lambda\mathcal{C}$ is homogeneous coordinates is the viewing ray), thus $\mathcal{C} \in RNS(M)$

Calibrated Cameras: Viewing Rays Coordinates

We can associate to each point on the image, a viewing ray

$$\mathbf{v} = \mathbf{C} + \lambda \begin{bmatrix} (KR)^{-1} \mathbf{p} \\ 0 \end{bmatrix}$$

Remember that $\mathbf{C} = RNS(M)$



Once Calibrated

We can associate to each point on the image, a viewing ray

$$\mathbf{v} = \mathbf{C} + \lambda \begin{bmatrix} (KR)^{-1} \mathbf{p} \\ 0 \end{bmatrix}$$

In fact

$$M\mathbf{v} = M \left(\mathbf{C} + \lambda \begin{bmatrix} (KR)^{-1} \mathbf{p} \\ 0 \end{bmatrix} \right)$$

$$M\mathbf{v} = MC + \lambda K[R, T] \begin{bmatrix} (KR)^{-1} \mathbf{p} \\ 0 \end{bmatrix}$$

$$M\mathbf{v} = \lambda [KR, KT] \begin{bmatrix} (KR)^{-1} \mathbf{p} \\ 0 \end{bmatrix}$$

$$M\mathbf{v} = \lambda \mathbf{p} = \mathbf{p} \quad \forall \lambda$$

Once Calibrated: Factorization of M

Once the camera has been calibrated we want to factorize M as

$$M = K[R, \mathbf{t}]$$

where R is a rotation matrix, and K is upper triangular.

By doing so we can:

- Compute viewing rays, which requires $(KR)^{-1}$
- Preserve K when the camera moves, since the intrinsic parameters do not change and do not need to be estimated from scratches.

Factorization of M

For computing viewing rays we need to factorize the matrix M estimated through camera calibration

$$M \simeq K[R, \mathbf{t}]$$

Let us write

$$M = [\tilde{M}, \mathbf{m}_4]$$

Then we have to find a way to write $\tilde{M} = KR$, where K is upper triangular and R is a rotation matrix.

Factorization of M

Let us compute the QR-decomposition of \tilde{M}^{-1} ($\tilde{M} \in \mathbb{R}^{3 \times 3}$)

$$\tilde{M}^{-1} = \tilde{Q} \tilde{T}$$

where \tilde{Q} is orthogonal and \tilde{T} is upper triangular. Since

$$\tilde{M}^{-1} = R^{-1} K^{-1},$$

Then we need to set

$$R = \det(\tilde{Q}) \tilde{Q}^T, \quad K = \tilde{T}^{-1}$$

Since by doing so $\det(R) > 0$ (we are allowed changing the sign of a matrix, thus we obtain a rotation and no reflection)

Similarly we can multiply $\frac{1}{K(3,3)}$ to have $\mathbf{1}$ in the bottom right entry of K

$$\mathbf{t} = \det(\tilde{Q}) K^{-1} \mathbf{m}_4$$

Factorization of M

```
function [K,R,t] = krt (M)
```

```
%KRT Internal and external parameters from M
```

```
[Q,T] = qr_p(inv(M(1:3, 1:3))); %qr decomposition  
yielding positive diagonal values over T, to make  
qr factorization unique
```

```
s = det(Q);
```

```
R = s*Q';
```

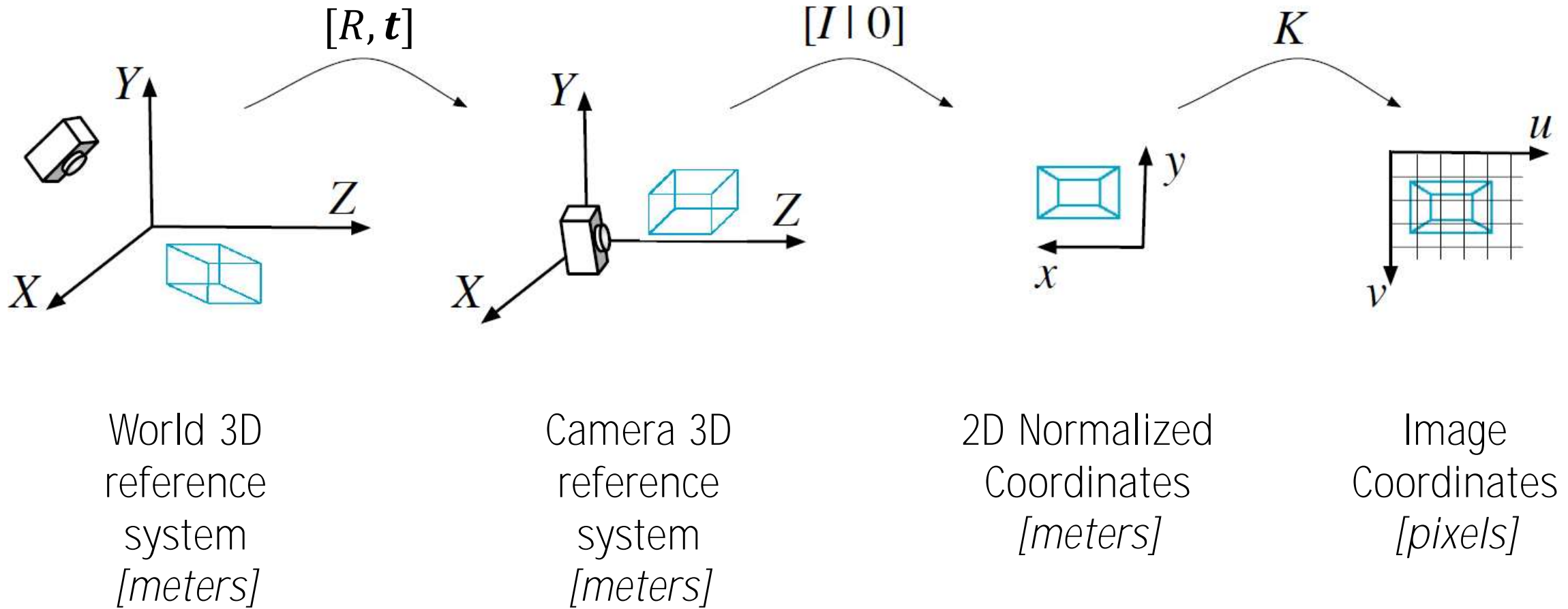
```
t = s*T*M(1:3,4);
```

```
K = inv(T./T(3,3));
```

```
end
```

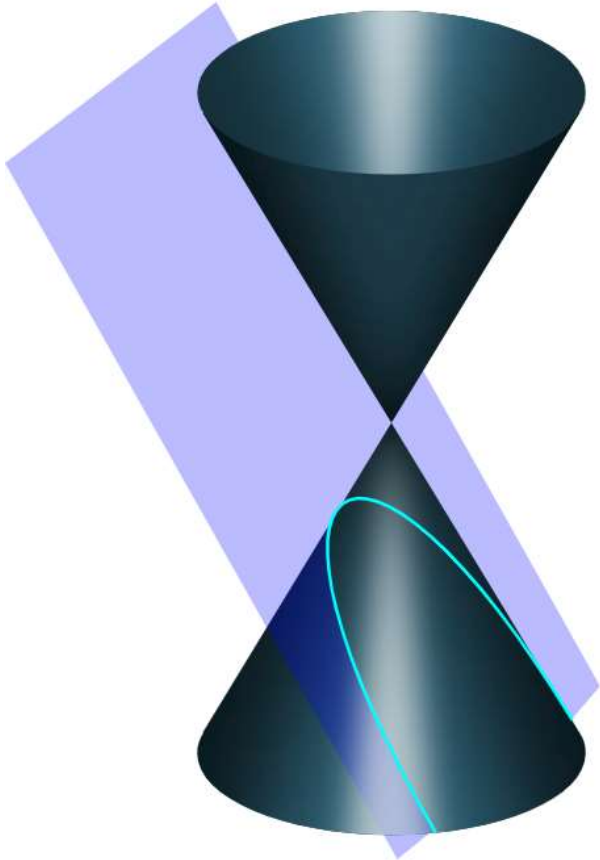
A recap on Camera Projection

The Coordinate Systems Involved in Camera Projection

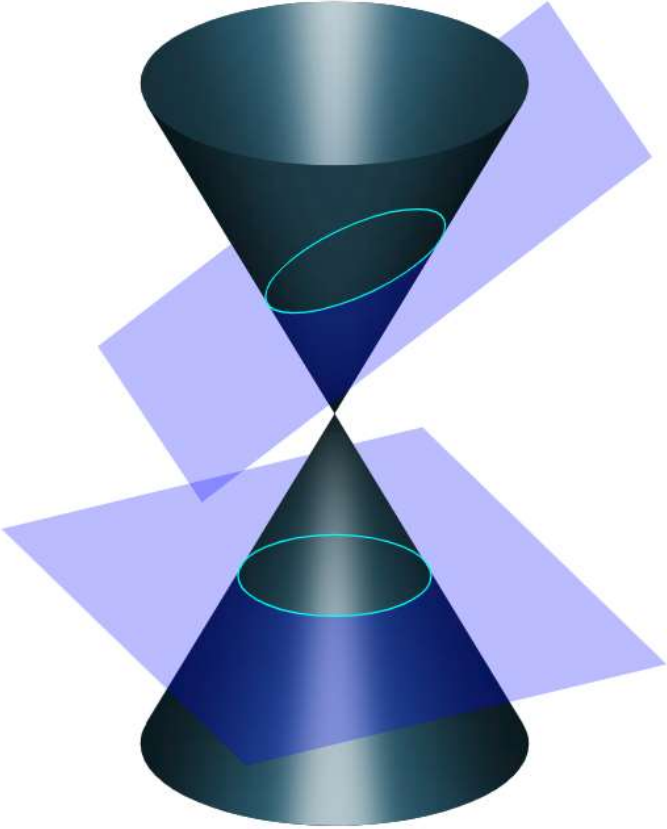


Conics

Conics



①



②



③

Conics in \mathbb{P}^2

A conic is a curve described by a second-degree equation in the plane.

In Euclidean coordinates a conic becomes

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

i.e. a polynomial of degree 2. “Homogenizing” this by the replacements:

$x \rightarrow x_1/x_3$, $y \rightarrow x_2/x_3$ gives

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

where the conic coefficient matrix \mathbf{C} is given by $\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

Conics in \mathbb{P}^2

$$\mathbf{x}^\top \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \mathbf{x} = 0$$

Rmk The conic coefficient matrix is symmetric,

Rmk multiplying C by a non-zero scalar does not change. Only the ratios of the elements in C are important, as for homogeneous points and for lines.

Rmk The conic has five degrees of freedom :

- the ratios $\{a : b : c : d : e : f\}$ or equivalently
- the six elements of a symmetric matrix less one for scale.

Conic Fitting

Conic fitting

Five points define a conic.

Let $\mathbf{x} = [x_i; y_i] \in \mathbb{R}^2$ a point belonging to a conic

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

To determine the conic coefficients we need five such equations in $\mathbf{c} = (a, b, c, d, e, f)'$, since the conic is determined up to a scaling factor.

In matrix expression

$$(x_i^2, \quad x_iy_i, \quad y_i^2, \quad x_i, \quad y_i, \quad 1)\mathbf{c} = 0$$

We re-arrange the first term as a matrix, the solution to solve is

Conic fitting

$$\begin{pmatrix} x_1^2, & x_1 y_1, & y_1^2, & x_1, & y_1, & 1 \\ & & \dots & & & \\ x_N^2, & x_N y_N, & y_N^2, & x_N, & y_N, & 1 \end{pmatrix} \mathbf{c} = \mathbf{0},$$

$$i. e., \quad A\mathbf{c} = \mathbf{0}$$

Rmk: The system is $A \in \mathbb{R}^{N \times 6}$, solved by $\mathbf{c} \in RNS(A)$ when $N = 5$

Unique solution, when $N = 4$ the RNS has dimension larger than one, any of the vectors in $RNS(P)$ is a solution

Conic fitting

$$\begin{pmatrix} x_1^2, & x_1 y_1, & y_1^2, & x_1, & y_1, & 1 \\ & & \dots & & & \\ x_N^2, & x_N y_N, & y_N^2, & x_N, & y_N, & 1 \end{pmatrix} \mathbf{c} = 0,$$

$$i. e., \quad A\mathbf{c} = 0$$

Rmk: The conic identified by \mathbf{c} is the same as the one from $\lambda\mathbf{c}$

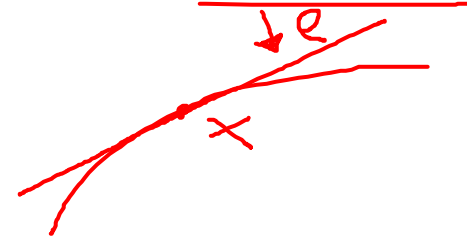
So, when $N > 5$ we are interested in solving

$$\mathbf{m}^* = \operatorname{argmin} \|\mathbf{A}\mathbf{m}\|_2 \text{ s. t. } \|\mathbf{m}\|_2 = 1$$

.... DLT!

Lines tangent to a conic

Theorem The line \mathbf{l} tangent to \mathbf{C} at a point \mathbf{x} on \mathbf{C} is given by $\mathbf{l} = \mathbf{C}\mathbf{x}$.



Proof (HZ result 2.7)

First, let's prove that the line $\mathbf{l} = \mathbf{C}\mathbf{x}$ passes through \mathbf{x} , since $\mathbf{l}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{C}\mathbf{x} = 0$. So, \mathbf{l} has at least one contact with the conic.

Now we prove that \mathbf{x} is the only point belonging to both \mathbf{l} and \mathbf{C} , thus \mathbf{l} is the tangent in \mathbf{x} .

Suppose that \mathbf{l} meets the conic in another point \mathbf{y} , such that $\mathbf{y}^\top \mathbf{C}\mathbf{y} = 0$ and $\mathbf{l}^\top \mathbf{y} = 0$. This also implies $\mathbf{l}^\top \mathbf{y} = (\mathbf{x}^\top \mathbf{C})\mathbf{y} = \mathbf{x}^\top \mathbf{C}\mathbf{y} = 0$. Then $(\mathbf{x} + \alpha\mathbf{y})^\top \mathbf{C}(\mathbf{x} + \alpha\mathbf{y}) = 0$ for all α , which means that the line $\mathbf{l} = \mathbf{C}\mathbf{x}$ joining \mathbf{x} and \mathbf{y} lies on the conic \mathbf{C} , which is therefore degenerate.

Matlab Example

Comments on Error Minimization

DLT and Error minimization

DLT algorithm minimizes the algebraic residual:

$$\mathbf{h}^* = \operatorname{argmin} \|\mathbf{A}\mathbf{h}\|_2 \text{ s.t. } \|\mathbf{h}\|_2 = 1$$

This does not have a physical meaning being \mathbf{A} (in case of homography estimation) made of rows like

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i \mathbf{x}_i^\top & y'_i \mathbf{x}_i^\top \\ w'_i \mathbf{x}_i^\top & \mathbf{0}^\top & -x_i \mathbf{x}_i^\top \end{bmatrix}$$

The norm of $\mathbf{A}\mathbf{h}$ does not admit a straightforward geometric interpretation

DLT and Error minimization

It would be better to measure the re-projection error, which has a physical meaning and refers to metric quantities (e.g. cm)

$$\frac{1}{N} \sum_i \|\mathbf{x}'_i - H\mathbf{x}_i\|_2 + \|\mathbf{x}_i - H^{-1}\mathbf{x}'_i\|_2$$

Provided the set of pairs $\{(\mathbf{x}'_i, \mathbf{x}_i), i = 1, \dots, N\}$

This indicates how far the mapped points are to the destination.

However, minimizing this error is non trivial.

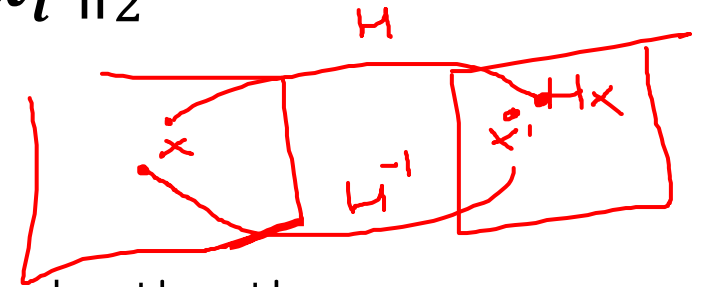
Rmk. If we take only 4 pairs of points, the algebraic and geometric error is zero

DLT and Error minimization

It would be better to measure the re-projection error, which has a physical meaning and refers to metric quantities (e.g. cm)

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This indicates how far the mapped points are to the destination.

However, minimizing this error is non trivial.

Rmk. If we take only 4 pairs of points, the algebraic and geometric error is zero

The standard procedure

1. Estimate the homography \hat{H} by minimizing the Algebraic error (e.g. through DLT)
2. Refine the estimated homography by minimizing the geometric error through a numerical solver
 - In this latter step, use \hat{H} as an initial guess for the nonlinear minimization

Dual Conic

Dual conics (conics envelopes)

Conics are defined by point equations

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

However, point-line duality makes it possible to define conics by lines

A line \mathbf{l} *tangent* to the conic \mathbf{C} satisfies

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$

where \mathbf{C}^* is the *adjoint* of \mathbf{C} which in case of symmetric and non-singular matrix is $\mathbf{C}^* = \mathbf{C}^{-1}$

Dual conics (conics envelopes)

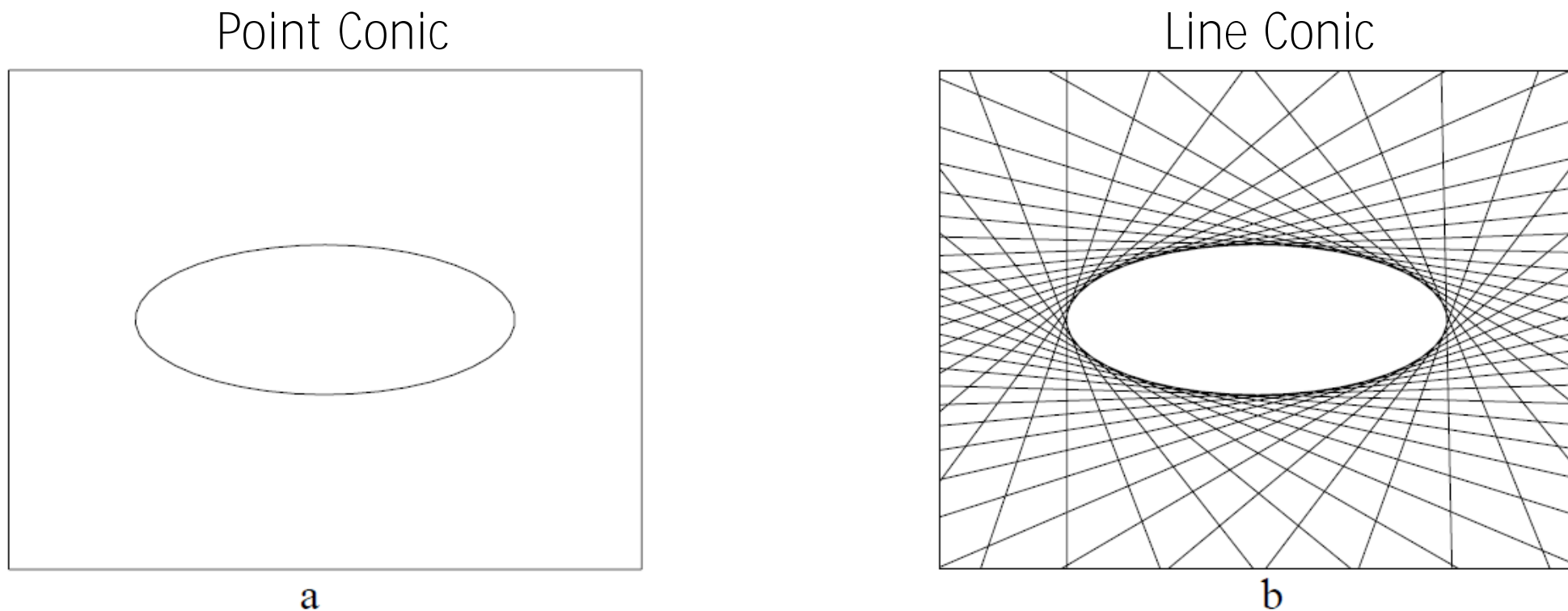


Fig. 2.2. (a) Points \mathbf{x} satisfying $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$ lie on a point conic. (b) Lines \mathbf{l} satisfying $\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$ are tangent to the point conic \mathbf{C} . The conic \mathbf{C} is the envelope of the lines \mathbf{l} .

Homographies and lines and conics

An homography transform each point \mathbf{x} in a point \mathbf{x}' such that:

$$\mathbf{x} \rightarrow H\mathbf{x} = \mathbf{x}'$$

An homography transform each line \mathbf{l} in a line \mathbf{l}' such that:

$$\mathbf{l} \rightarrow (H^{-1})^{\top} \mathbf{l} = \mathbf{l}'$$

An homography transform each conic \mathcal{C} in a conic \mathcal{C}' such that:

$$\mathcal{C} \rightarrow (H^{-1})^{\top} \mathcal{C} H^{-1} = \mathcal{C}'$$

An homography transform each dual (line) conic \mathcal{C}^* in a dual conic \mathcal{C}^{*}

$$\mathcal{C}^* \rightarrow H\mathcal{C}^*H^{\top} = \mathcal{C}^{*}$$

We say that points transform *contravariantly* and lines and conics transform *covariantly*.

Degenerate Conics

When the matrix \mathbf{C} is not of full rank, a point conic \mathbf{C} degenerates in

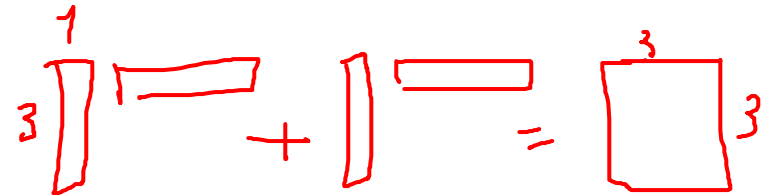
- two lines (\mathbf{C} has rank **2**),
- or a repeated line (\mathbf{C} has rank **1**)

A line conic \mathbf{C}^* degenerates in

- two points (\mathbf{C}^* has rank **2**),
- a repeated point (\mathbf{C}^* has rank **1**)

For instance, \mathbf{C}^* such that $\text{rank}(\mathbf{C}^*) = 2$

$$\mathbf{C}^* = \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$$



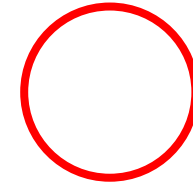
Is the set of all the lines passing through \mathbf{x} and \mathbf{y} (not the that \mathbf{C}^* remains a 3×3 matrix)

Circular Points and Conic dual to circular points

Circular Points: I and J

These are very special points in l_∞

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$



These are called circular points since they correspond to the intersection of any circle with l_∞

Rmk a circle can be fit through 3 points. Or can be fit as a general conic passing through 3 points + the two circular points

Circular Points: ***I*** and ***J***

Circular points are fixed under an orientation-preserving similarity

$$H_S \mathbf{I} = \mathbf{I}, \quad H_S \mathbf{J} = \mathbf{J}$$

a reflection instead swaps ***I*** and ***J***

Rmk circular points are not fixed under projective transformations

Conic dual to circular points: \mathcal{C}_∞^*

Let's take \mathcal{C}^*

$$\mathcal{C}^* = \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$$

For which $\text{rank}(\mathcal{C}^*) = 2$ and is the set of all the lines passing through \mathbf{x} and \mathbf{y} (note that \mathcal{C}^* remains a 3×3 matrix)

We can compute the dual conic associated to circular points

$$\mathcal{C}_\infty^* = \mathbf{I}\mathbf{J}^\top + \mathbf{J}\mathbf{I}^\top$$

which also is a degenerate line conic

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \rightarrow \mathcal{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Conic dual to circular points: \mathcal{C}_∞^*

Property \mathcal{C}_∞^* is fixed under an orientation-preserving similarity

Property \mathcal{C}_∞^* in the Euclidean plane can be written as

$$\mathcal{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

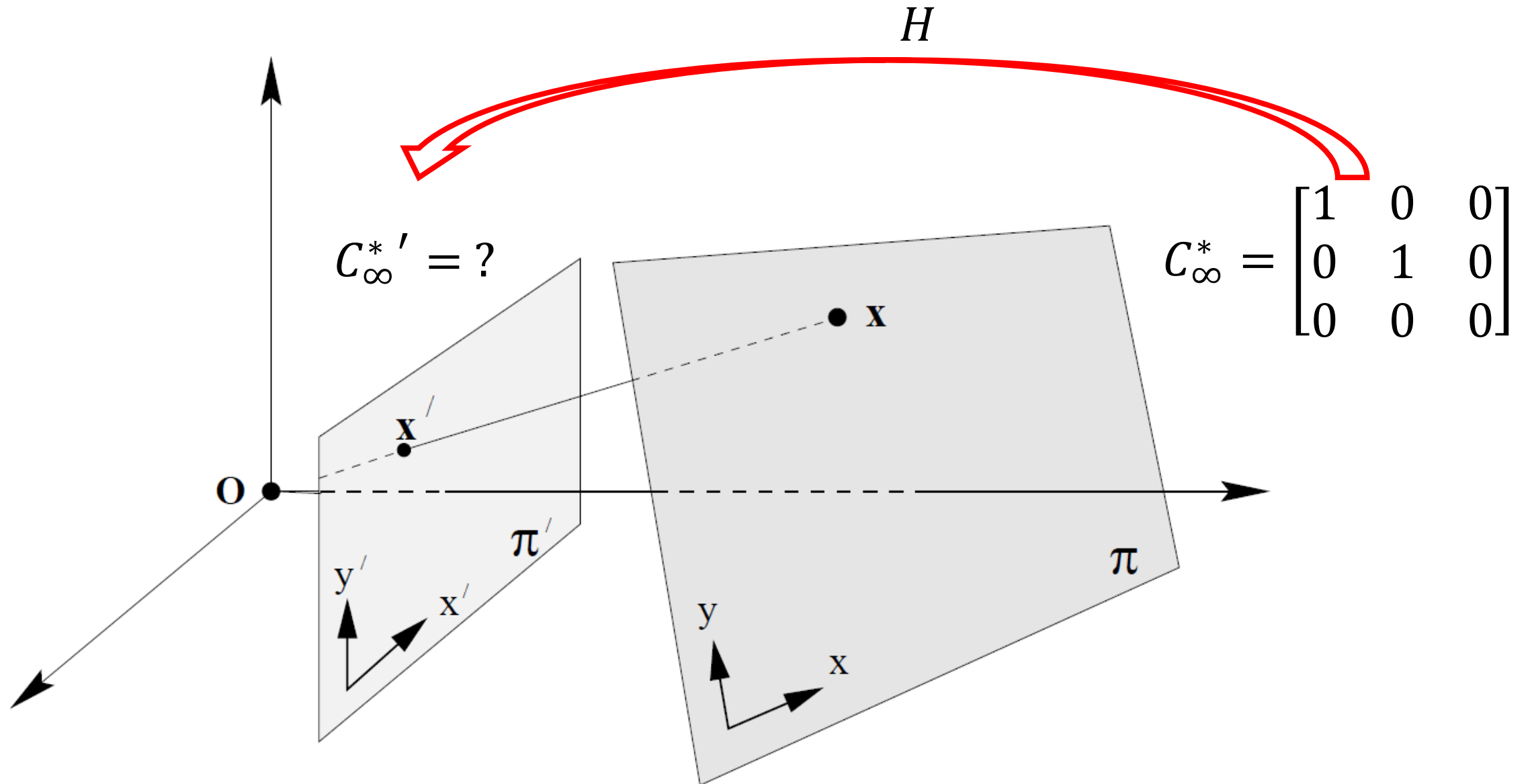
Therefore, any homography transforms \mathcal{C}_∞^* like any other line conic

$$\mathcal{C}_\infty^* \rightarrow \mathcal{C}_\infty^{*'} = H\mathcal{C}_\infty^*H^\top$$

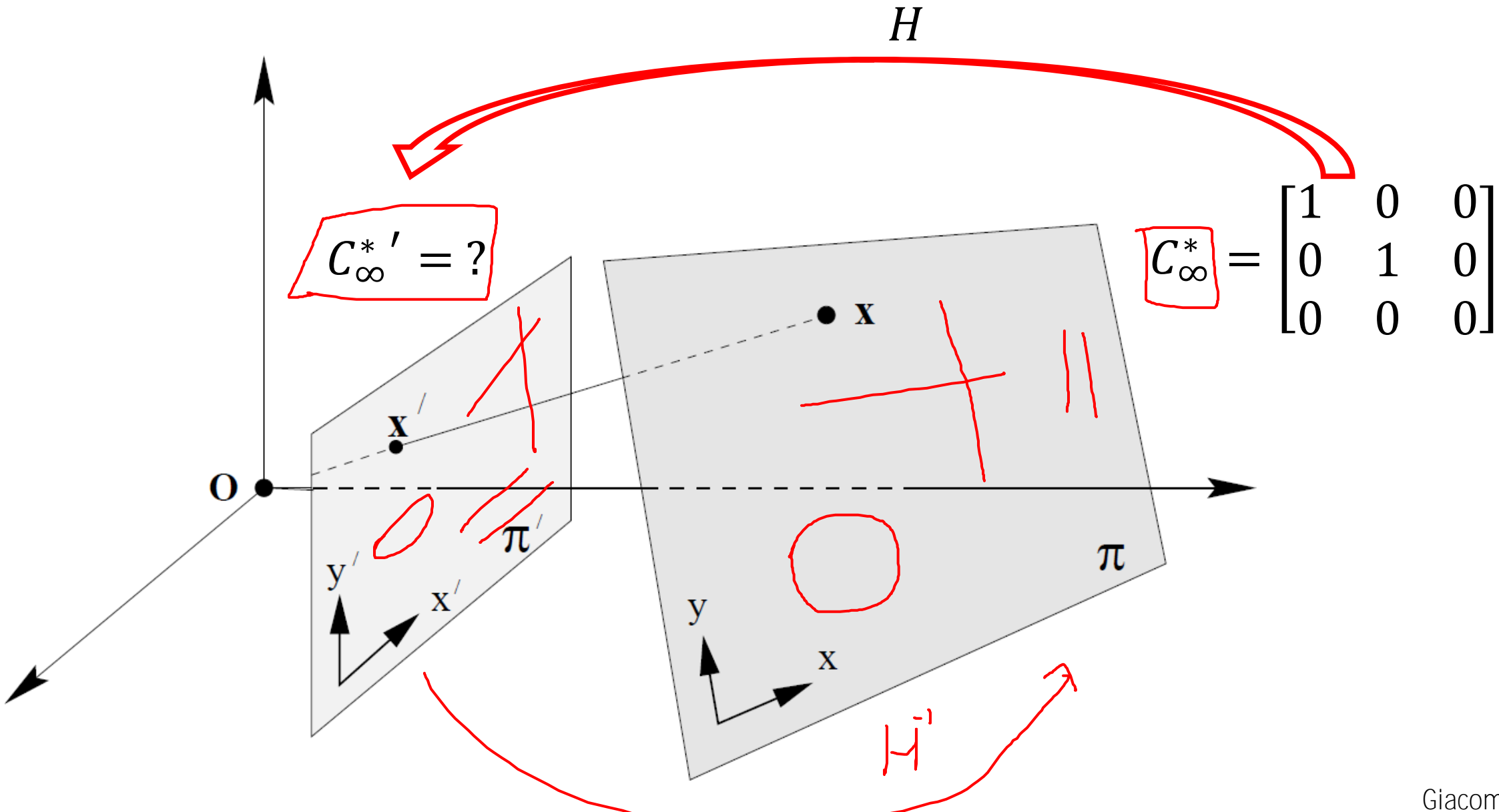
In an image, the equations of $\mathcal{C}_\infty^{*'}$ might be different from $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Single Image Rectification From \mathcal{C}_{∞}^*

Single Image Rectification



Single Image Rectification



Single Image Rectification

Affine rectification: find the homography H_P that sends l_∞ in its canonical equation $[0; 0; 1]$.

Metric rectification: after applying H_P , determine an affine transformation H_A that maps circular points back to their canonical position $[1; \pm i; 0]$.

- Any similarity transformation applied next would not change the location of circular points

Rmk it is enough to find an homography that brings C_∞^* in $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Single Image Rectification

Important Property after an affine rectification, $\mathcal{C}_{\infty}^{* \prime}$ can be written as

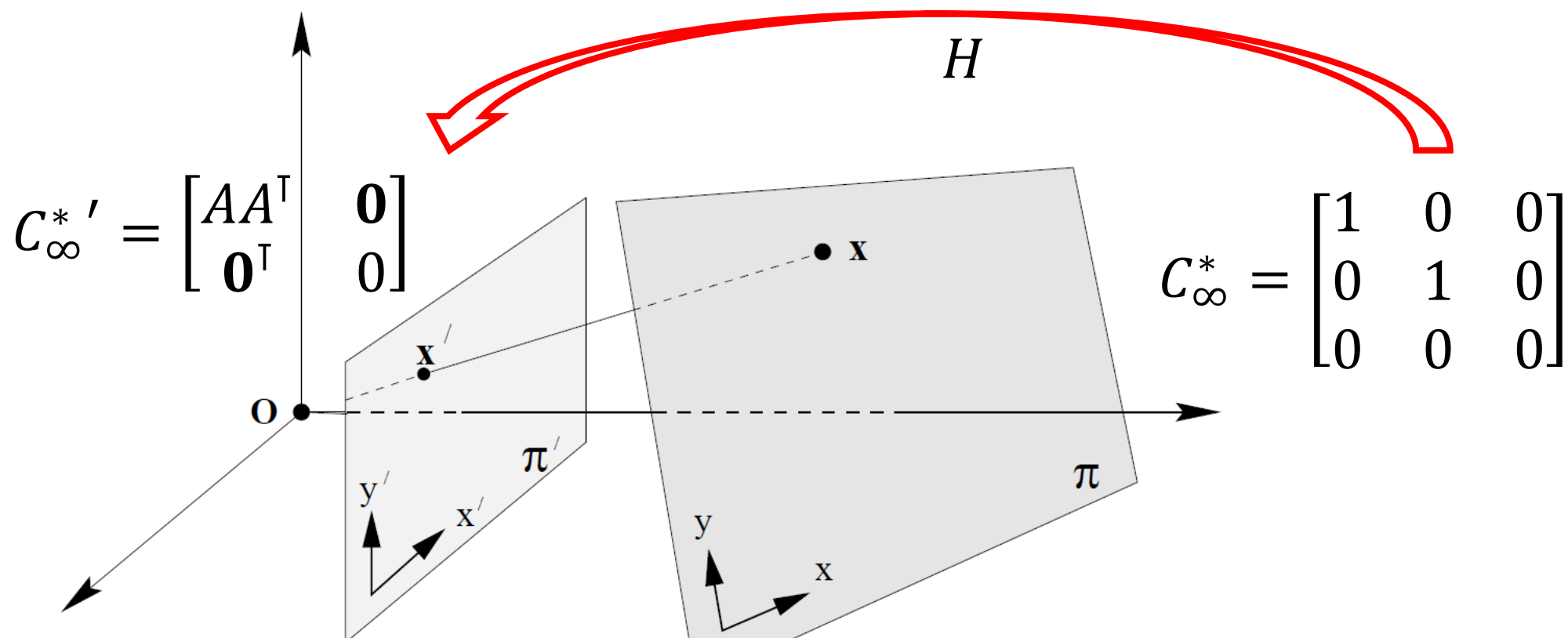
$$\mathcal{C}_{\infty}^{* \prime} = \begin{bmatrix} AA^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

Being A a 2×2 invertible matrix (cfr HZ example 2.26)

(this is the image of the canonical \mathcal{C}_{∞}^* through an affine transformation since similarities do not change \mathcal{C}_{∞}^* , and that projective transformation has been removed by the affine rectification)

Single Image Rectification

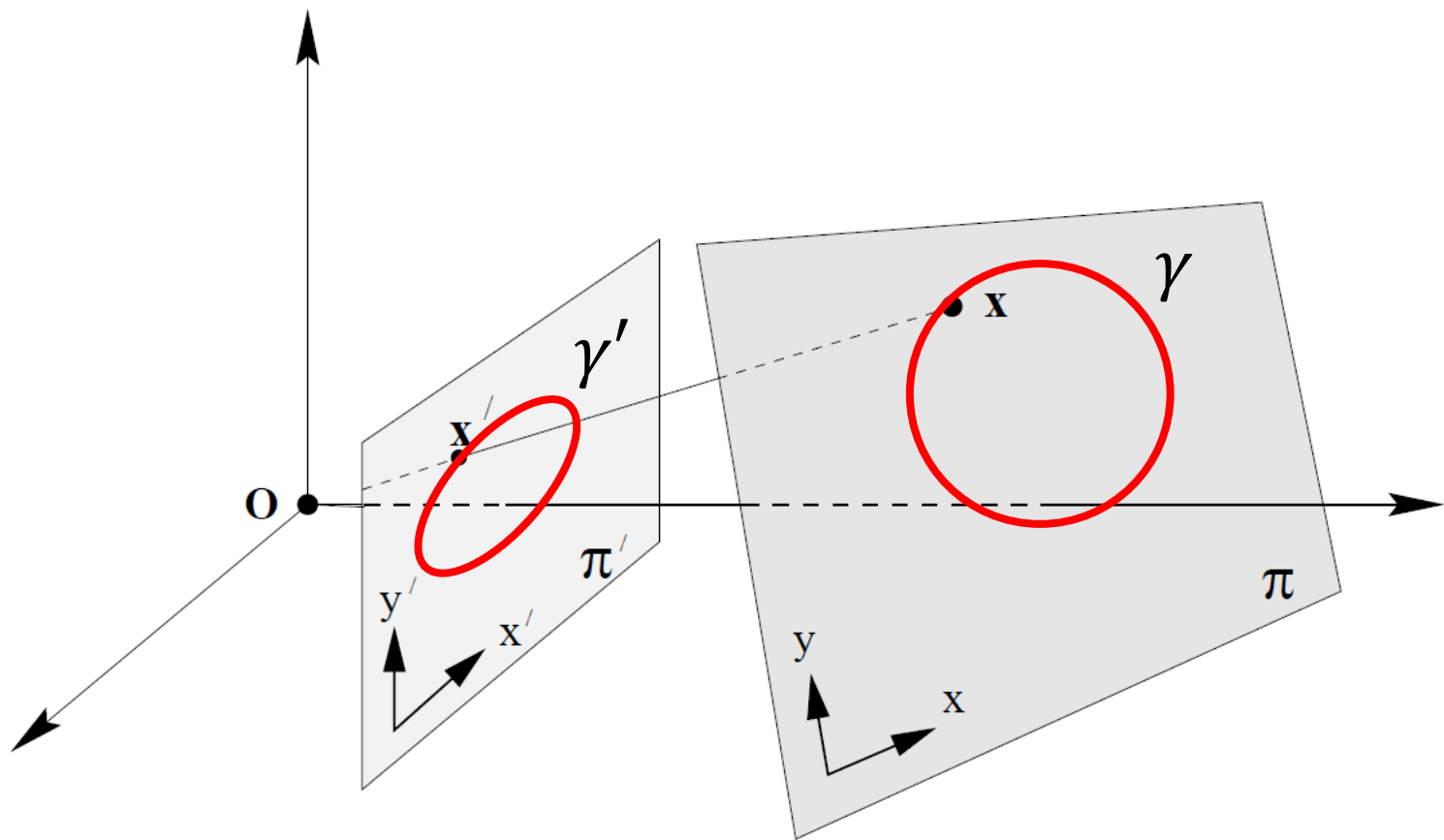
Our goal is to estimate that H in order to define the inverse mapping H^{-1}



Rectification can be achieved by estimating $C_{\infty}^{*'}$ (in this case AA^T) in the image

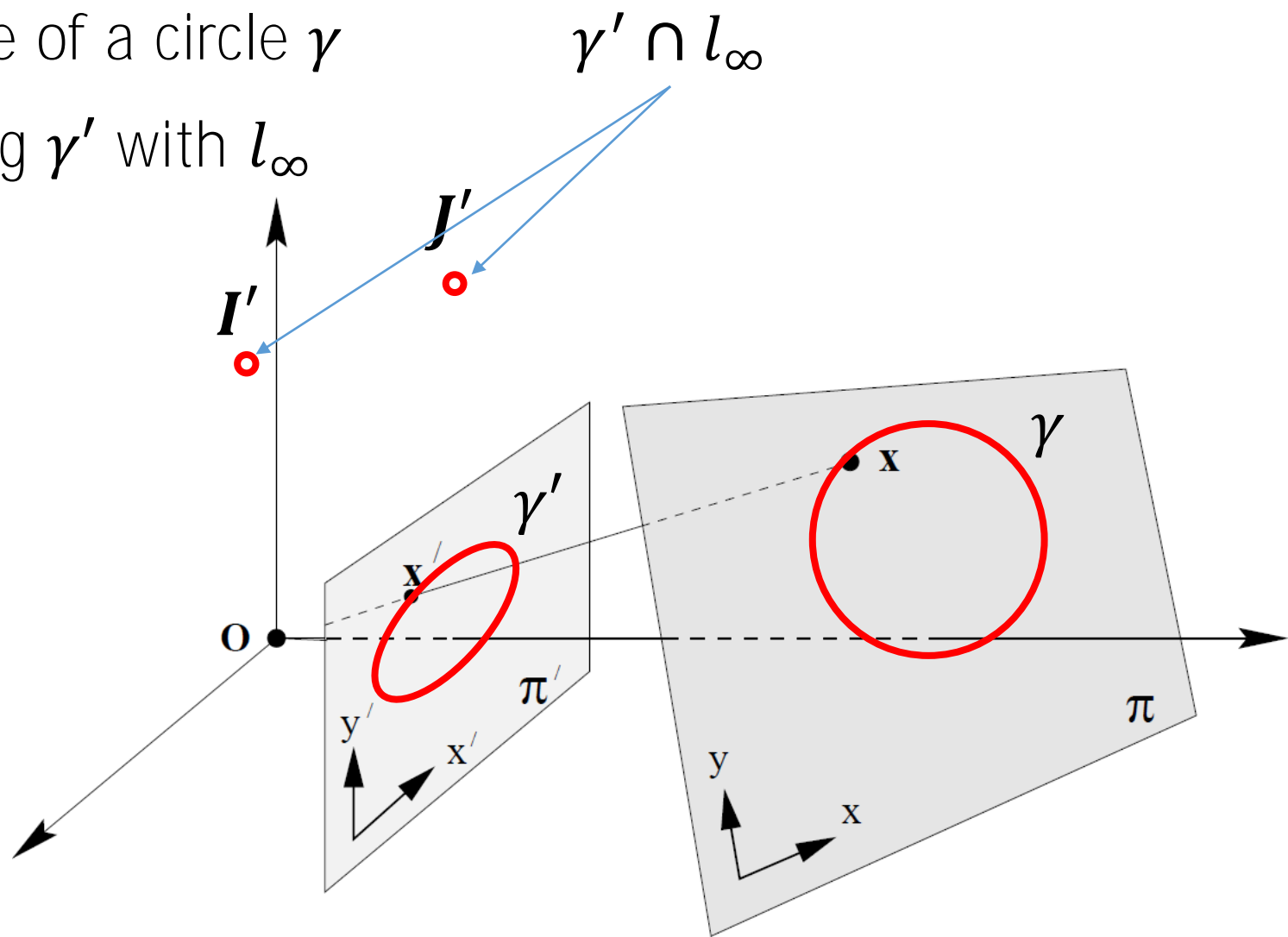
Single Image Rectification provided $\mathcal{C}_\infty^{*'}$

1. Perform affine rectification, such that $l_\infty = [0; 0; 1]$
2. Fit an ellipse to γ' the image of a circle γ
3. Compute \mathbf{I}', \mathbf{J}' by intersecting γ' with l_∞
4. We have $\mathcal{C}_\infty^{*'} = \mathbf{I}'\mathbf{J}'^\top + \mathbf{J}'\mathbf{I}'^\top$



Single Image Rectification provided $\mathcal{C}_\infty^{*'}$

1. Perform affine rectification, such that $l_\infty = [0; 0; 1]$
2. Fit an ellipse to γ' the image of a circle γ
3. Compute \mathbf{I}', \mathbf{J}' by intersecting γ' with l_∞
4. We have $\mathcal{C}_\infty^{*'} = \mathbf{I}'\mathbf{J}'^\top + \mathbf{J}'\mathbf{I}'^\top$



Single Image Rectification provided \mathcal{C}_∞^* '

The SVD of \mathcal{C}_∞^* ' can be used to define the rectifying homography transformation.

In fact, up to a similarity transformation that rescales eigenvalues we obtain \mathcal{C}_∞^* !

$$SVD(\mathcal{C}_\infty^*) = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

So, it is enough to take U to rectify the image, since the similarity transformation does not affect \mathcal{C}_∞^*

Single Image Rectification From Orthogonal Lines

Recap: Angles in the Euclidean Plane

In Euclidean geometry the angle between two lines is computed from the dot product of their normals. For the lines $\mathbf{l} = (l_1; l_2; l_3)$, and $\mathbf{m} = (m_1; m_2; m_3)$, the angle θ is such that

$$\cos \theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

We will see later interesting properties on projective plane

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Time has come!

Rmk the first two components of \mathbf{l} and \mathbf{m} do not have a pre-defined meaning in \mathbb{P}^2

Angles in the Projective Plane

An analogous expression, that is invariant to projective transformation, for the angle θ between lines $\mathbf{l} = (l_1; l_2; l_3)$ and $\mathbf{m} = (m_1; m_2; m_3)$, is

$$\cos \theta = \frac{\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^T \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^T \mathbf{C}_\infty^* \mathbf{m})}}$$

Rmk when $\mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and we are in homogeneous coordinates. The system the above equation becomes the same as in the previous slide

Angles in the Projective Plane

Rmk it is easy to verify that

$$\cos \theta = \frac{\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^\top \mathbf{C}_\infty^* \mathbf{l})(\mathbf{m}^\top \mathbf{C}_\infty^* \mathbf{m})}}$$

is invariant w.r.t. any projective transformation H , thus it's a perfect definition for an angle

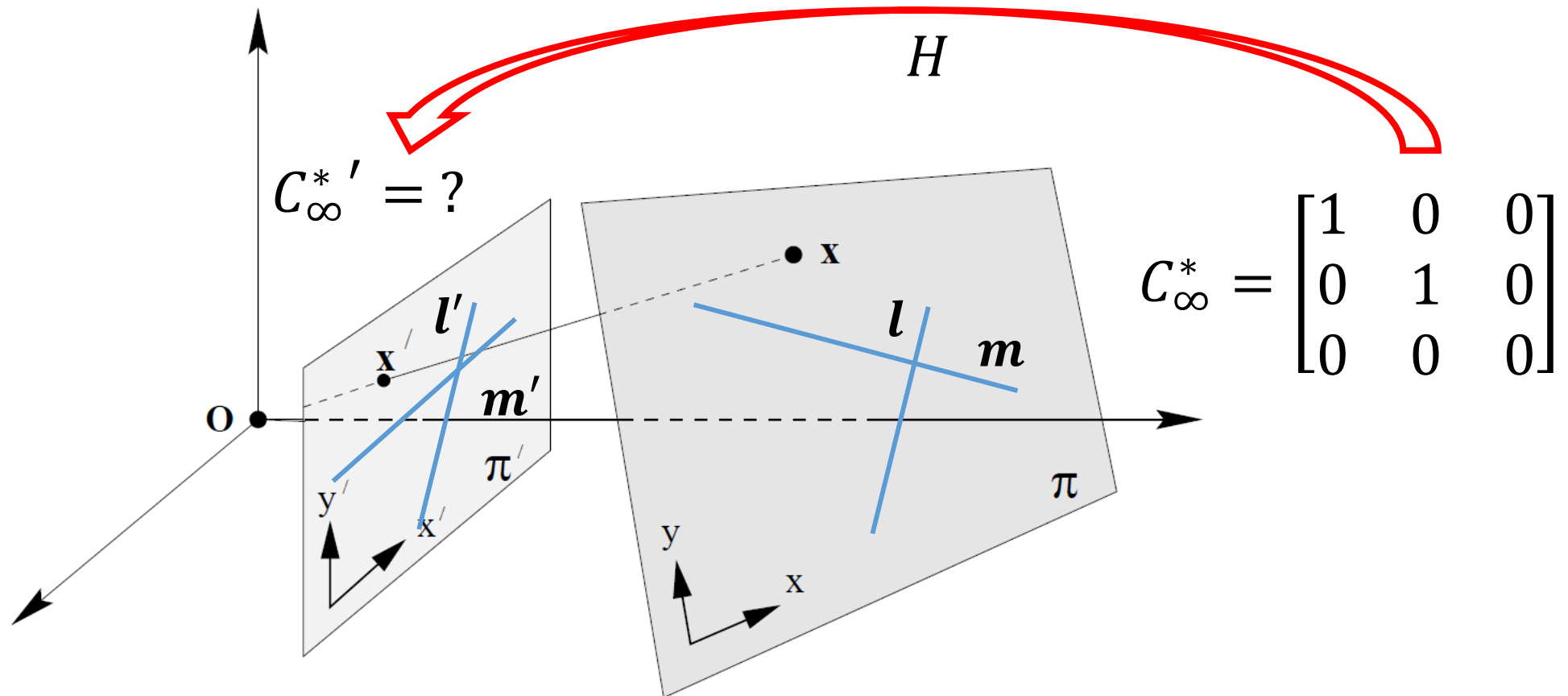
(just replace $\mathbf{l} \rightarrow (H^{-1})^\top \mathbf{l}$, and $\mathbf{C}^* \rightarrow H \mathbf{C}^* H^{-1}$)

Rmk the image of two orthogonal lines \mathbf{l}', \mathbf{m}' have to satisfy

$$\mathbf{l}'^\top \mathbf{C}_\infty^{*'} \mathbf{m}' = \mathbf{0}$$

Single Image Rectification from orthogonal lines

Let l', m' be the images of two orthogonal lines l and m in the 3D space



Single Image Rectification from orthogonal lines

If \mathbf{l}' and \mathbf{m}' are images two orthogonal lines \mathbf{l} and \mathbf{m} (in the 3D world), then it has necessarily to hold

$$\mathbf{l}'^T \mathbf{C}_{\infty}^{*'} \mathbf{m}' = 0$$

Let us use this information to compute \mathbf{C}_{∞}^{*}' , thus compute H

Remember that, after an affine rectification

$$\mathbf{C}_{\infty}^{*}' = \begin{bmatrix} \mathbf{A}\mathbf{A}^T & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

and that $\mathbf{S} = \mathbf{A}\mathbf{A}^T$ is a symmetric homogeneous matrix ($\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$), thus there are only **2** unknowns to identify \mathbf{C}_{∞}^{*}'

Each pair of orthogonal lines yield a single equation

$$\mathbf{l}'(1:2)^T \mathbf{S} \mathbf{m}'(1:2) = 0$$

Single Image Rectification from orthogonal lines

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Let us **Two pairs of orthogonal lines are enough to identify $\mathbf{C}_\infty^{*'}$**

Remember that, after an affine rectification

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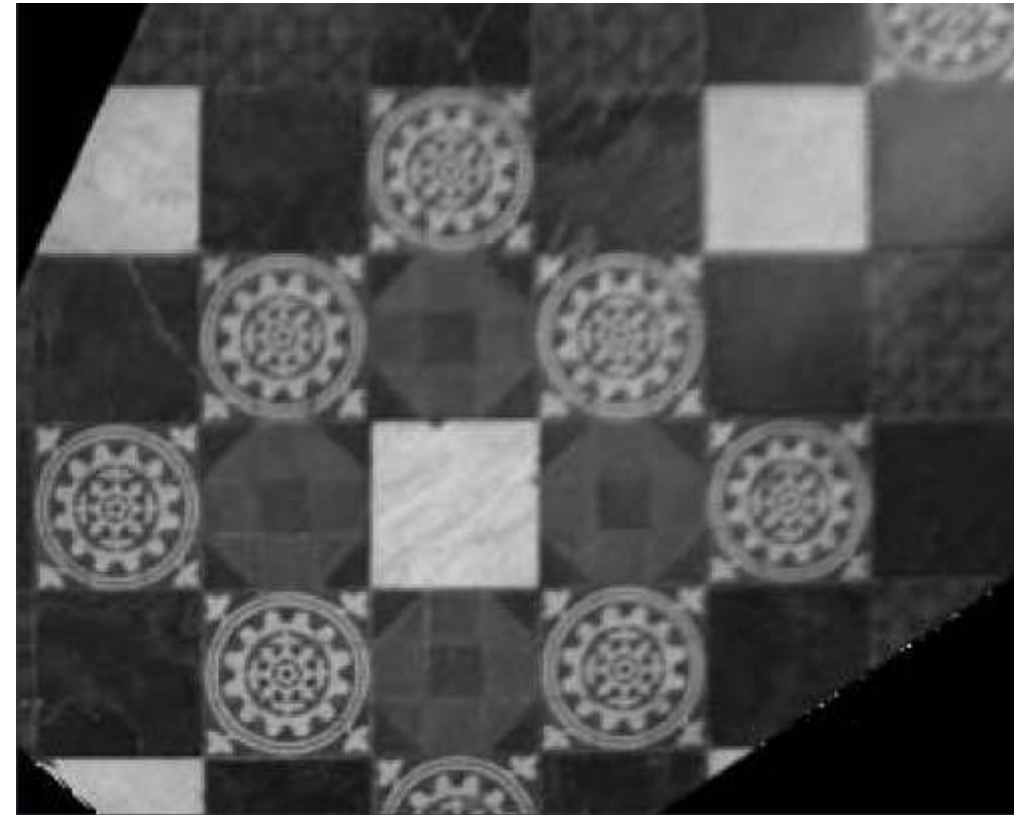
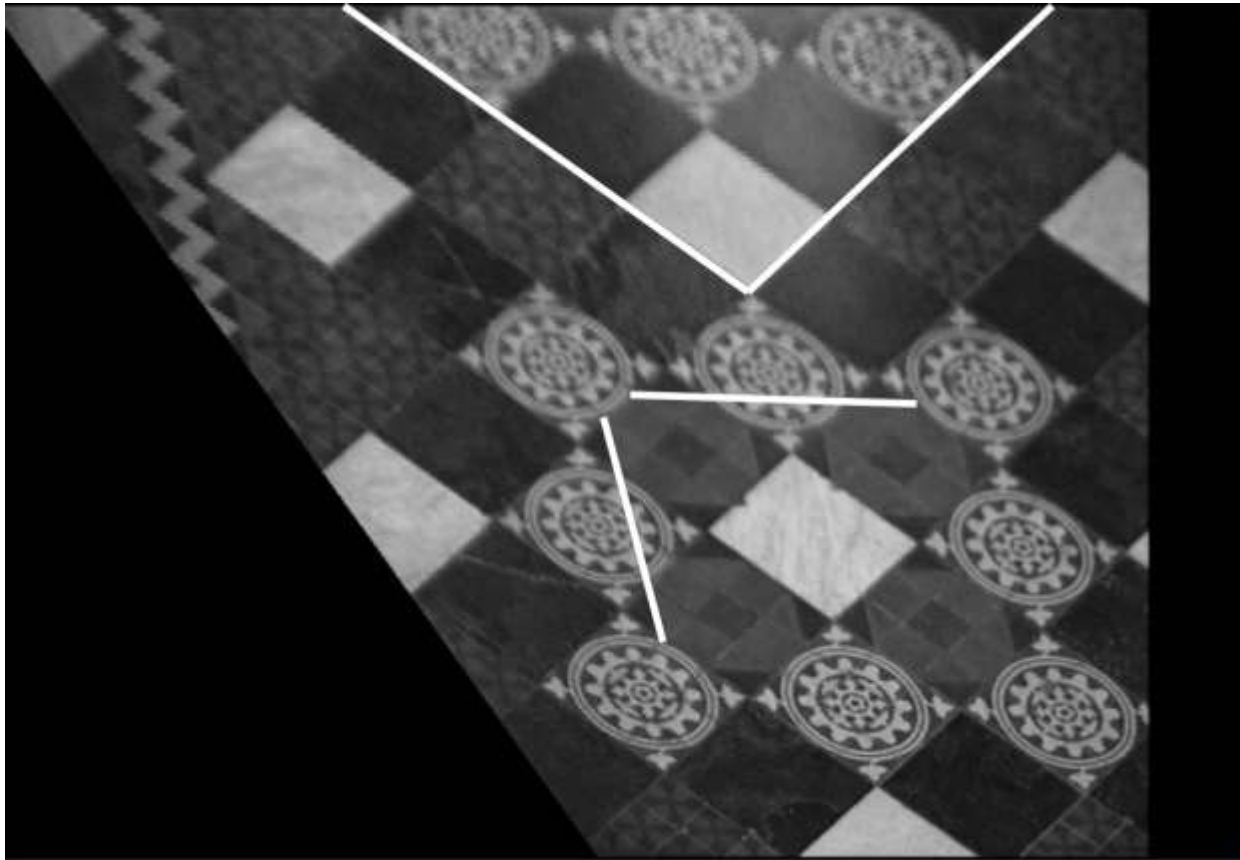
Each pair of orthogonal lines yield a single equation

$$\mathbf{l}'(1:2)^\top \mathbf{S} \mathbf{m}'(1:2) = 0$$

Stratified Rectification from Orthogonal Lines

$$\mathbf{l}'(1:2)^T \mathbf{S} \mathbf{m}'(1:2) = 0$$

Two pairs of (different, i.e. not parallel) orthogonal lines are enough to estimate \mathbf{S} , (thus \mathbf{A} through Choleski decomposition)



Rectification from 5 pairs of orthogonal lines

Assume no affine rectification has been done before

In principle, any pair of lines \mathbf{l}' , \mathbf{m}' that in the scene are orthogonal provides a single equation

$$\mathbf{l}'^T \mathbf{C}_\infty^* \mathbf{m}' = 0$$

This is a linear constraint in the entries of \mathbf{C}_∞^* (this is exactly like fitting a conic through 5 points).

Five of these equations are enough to set up a linear system

$$A\mathbf{c} = \mathbf{0}$$

where \mathbf{c} contains the six parameters of the conic. The solution is:

$$\mathbf{c} = RNS(A)$$

Rectification from 5 pairs of orthogonal lines

Now the pairs of lines (highlighted by solid white lines) can also be parallel in the 3D world (like the windows here), since affine rectification has not been performed, yet.



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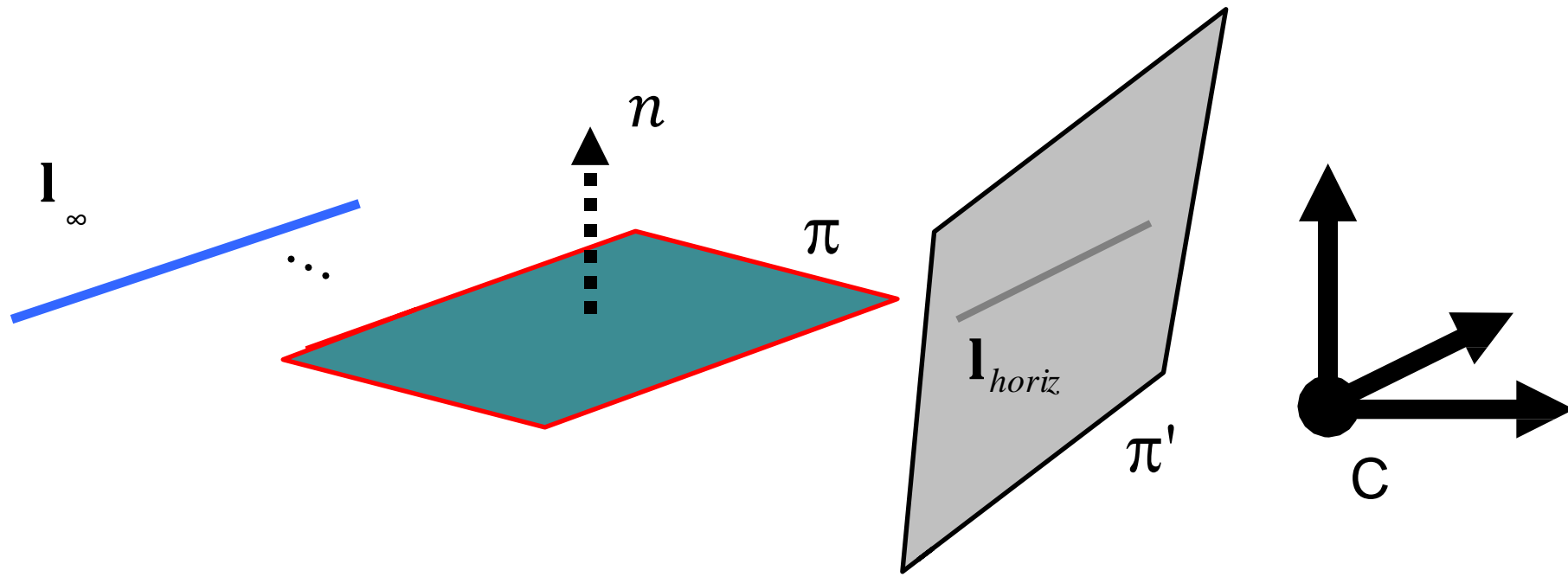


Camera Calibration and the 3D world

Normal to the horizon

Let $M = K[R, \mathbf{t}]$ be the projection matrix of a calibrated camera, the normal \mathbf{n} to the plane of an horizon l_h in the image is:

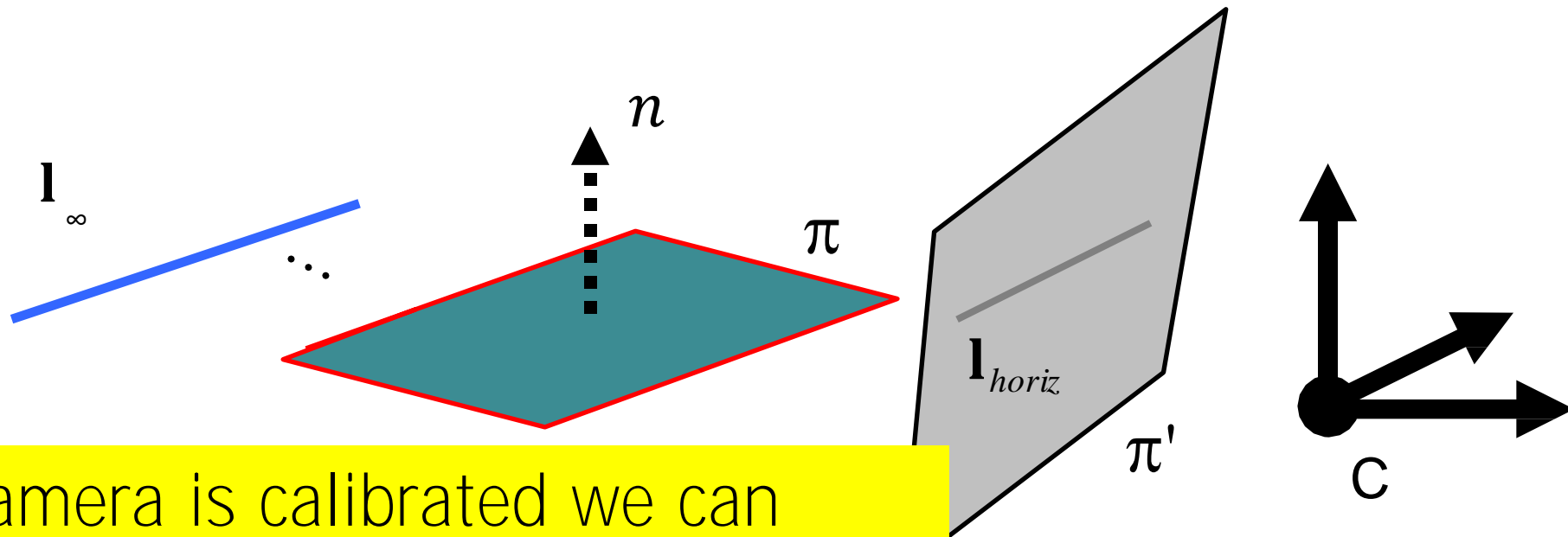
$$\mathbf{n} = K^T l_h$$



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$$\mathbf{n} = K^T l_h$$



Once the camera is calibrated we can compute the 3D orientation of any plane for which we see the horizon

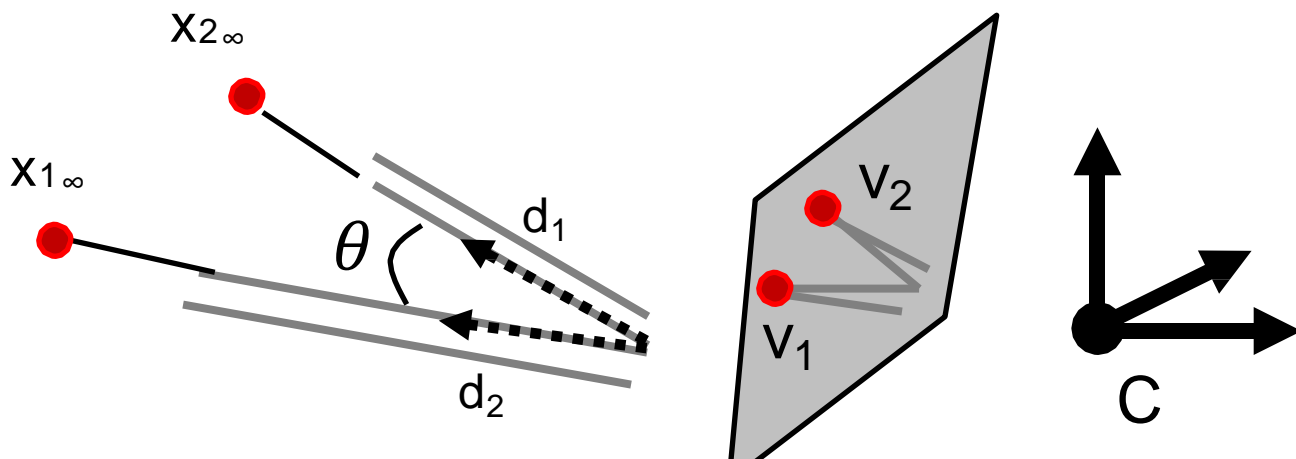
Angle between vanishing points

Let v_1 and v_2 be the vanishing points of two directions d_1 and d_2 , then

$$\cos(\theta) = \frac{d_1 \cdot d_2}{\|d_1\| \|d_2\|}$$

And the same can be also computed from the vanishing points

$$\cos(\theta) = \frac{v_1^T \omega v_2}{\sqrt{v_1^T \omega v_1} \sqrt{v_2^T \omega v_2}}$$



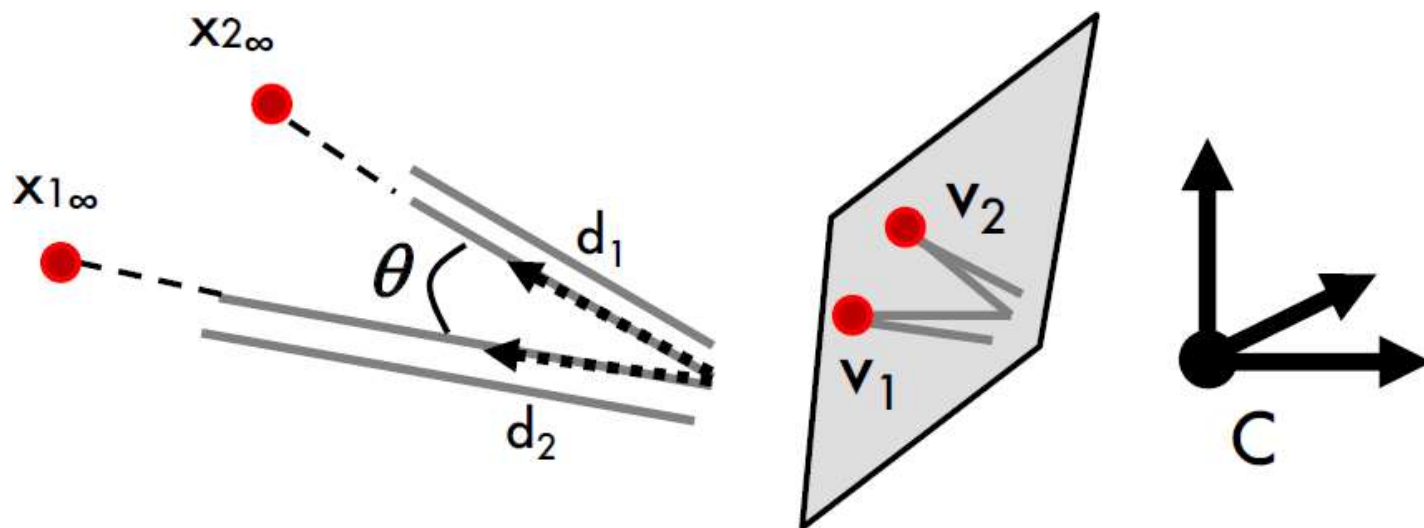
Angles between two vanishing points

Let v_1 and v_2 be the vanishing points of two lines (having ideal points $X_{1\infty}$ and $X_{2\infty}$), and let

$$\omega = (KK^T)^{-1}$$

Then if θ is the angle between the two line directions, then

$$\cos \theta = \frac{v_1^T \omega v_2}{\sqrt{v_1^T \omega v_1} \sqrt{v_2^T \omega v_2}}$$



Angles between two planes

Let l_1 and l_2 be the horizon of planes having normal n_1 and n_2

Then, the angle between the planes is

$$\cos(\theta) = \frac{n_1 \cdot n_2}{\|n_1\| \|n_2\|}$$
$$\cos \theta = \frac{l_1^\top \omega^{-1} l_2}{\sqrt{l_1^\top \omega^{-1} l_1} \sqrt{l_2^\top \omega^{-1} l_2}}$$

Camera Calibration from Vanishing Points

Vanishing points are images of points at infinity, and provide orientation information as fixed stars.

Ideal points are part of the scene and can be used as references. Their position in the image (i.e. the vanishing points) depend only on the camera rotation.

Calibration From 3D Orthogonal Vanishing Points

Note that any pair of vanishing point v_1, v_2 corresponding orthogonal lines in the image yields a scalar equation in ω

$$\frac{v_1^\top \omega v_2}{\sqrt{v_1^\top \omega v_1} \sqrt{v_2^\top \omega v_2}} = 0$$
$$v_1^\top \omega v_2 = 0$$

The matrix ω is symmetric ($\omega = (KK^\top)^{-1}$), thus 5 unknowns

$$\omega = \begin{bmatrix} w_1 & w_2 & w_4 \\ w_2 & w_3 & w_5 \\ w_4 & w_5 & w_6 \end{bmatrix}$$

If the camera has zero skew $w_2 = 0$

If pixels are squared $w_3 = w_1$

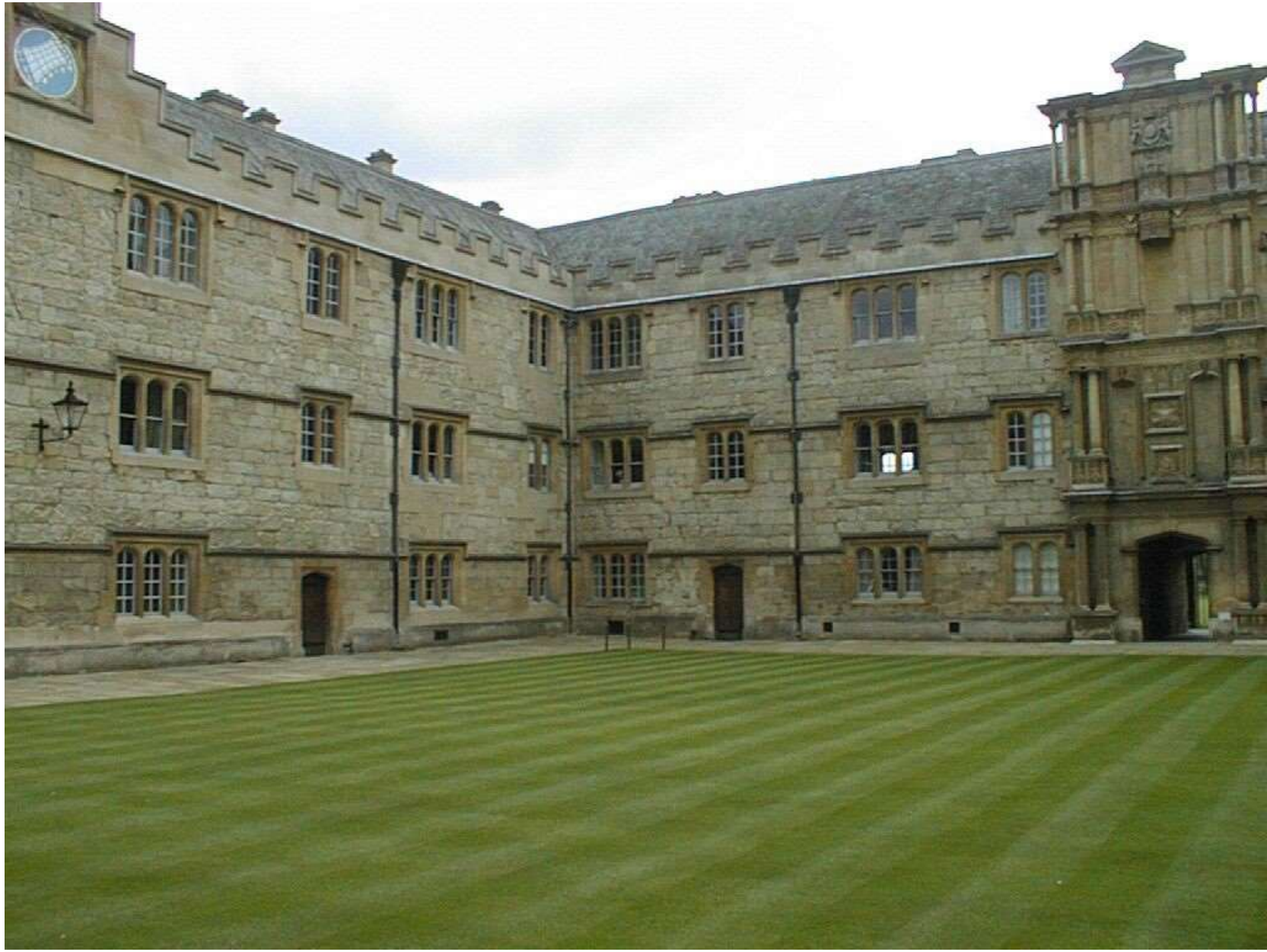
Calibration From 3D Orthogonal Vanishing Points

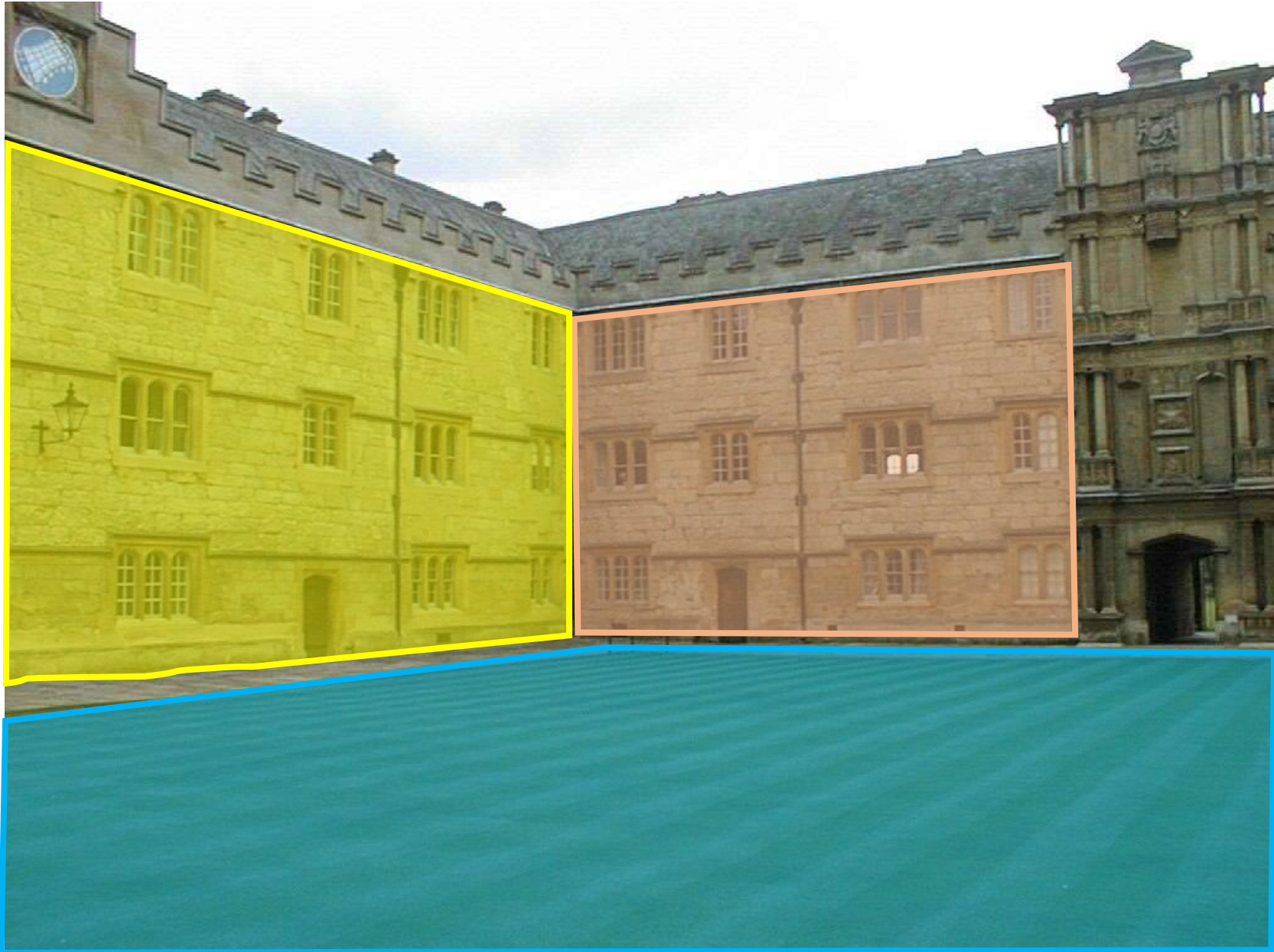
The matrix ω is symmetric for a squared pixel and zeros-skew camera

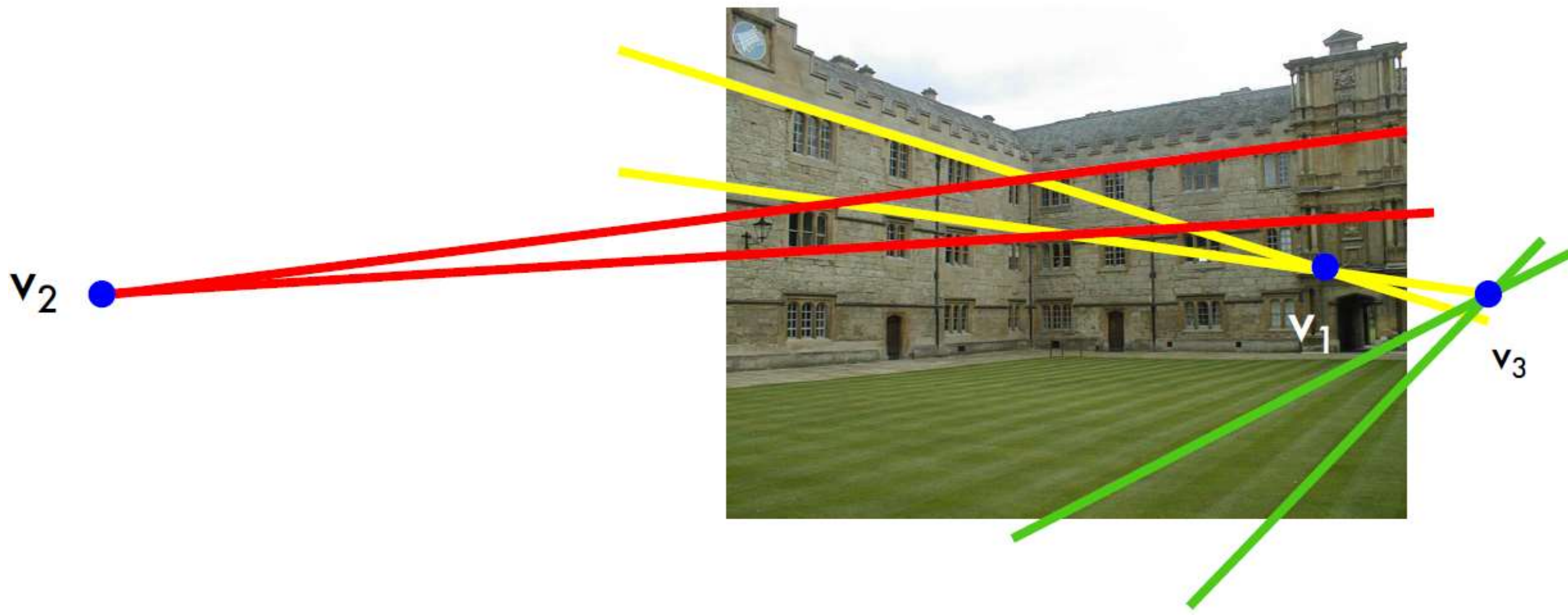
$$\omega = \begin{bmatrix} w_1 & 0 & w_4 \\ 0 & w_1 & w_5 \\ w_4 & w_5 & w_6 \end{bmatrix}$$

This has 4 unknowns (up to a scalar factor) -> three pairs of orthogonal lines can be used to compute ω

Once ω has been computed, K can be obtained by Cholesky factorization







Recap

- Implications of image calibration
 - Camera Center
 - Viewing Rays
 - Factorization of M
- Conics and Conic Fitting
 - Comments on Error Minimization
- Measuring Angles
- Single View Rectification

Next Class, Epipolar Geometry

